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NATIONAL CERTIFICATE MATHEMATICS

VOLUME II

(SECOND YEAR COURSE)

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PREFACE

THIS work is intended to provide a systematic and progressive text-book in mathematics for students taking mechanical or electrical engineering courses in a Technical Institution. It is in three volumes, which are planned to correspond to work which is usually done in the first three years of the senior course. The books include such mathematics as would normally be taken in a Technical Institution in which the students are preparing for a National Certificate. They also cover the syllabuses for the Examinations (S_1, S_2 and S_3) in practical mathematics conducted by the Union of Lancashire and Cheshire Institutes, the Union of Educational Institutions and the Northern Counties Technical Examination Council.

The practical requirements of technical students have been carefully borne in mind throughout the volumes, but an endeavour has also been made, within the limits necessarily imposed upon such a work, to provide a fundamental and theoretical basis such as is necessary for a more advanced study of the subject.

To meet the requirements of some syllabuses, certain topics, such as the Binomial Theorem, have been briefly introduced, but a full treatment is reserved for Vol. III.

The authors acknowledge, with gratitude, the permission which has been kindly given by the Union of Lancashire and Cheshire Institutes, the Union of Educational Institutions, the Northern Counties Technical Examinations Council, and the Board of Education, to use questions which have appeared in their Examinations, and by Messrs. Longmans Green and Company, Limited, to reproduce tables from " Mathematical Tables and Formulae."

P. A.
H. M.

GENERAL EDITOR'S FOREWORD

TECHNICAL Education stands on the verge of a great advance, probably the greatest in its history. Throughout the country there is general recognition of the necessity for a great expansion of the facilities which are available for thorough and systematic training for science and industry. This desire finds expression in the provision which has been made in the new Education Act for all forms of Technical Education. During the war many thousands of men and women have been compelled to undergo brief and hastily improvised courses in scientific and technical processes, and this has helped to bring home to them, and to most people, the importance of systematic technical education in the conditions of modern life.

There is general recognition of the well-founded conclusions of experts that the economic prosperity of this country will, more than ever before, be dependent upon the efficiency and adaptability of our system of scientific and technical training.

It is the hope and ambition of those who have been planning this series of technical books that it will meet the demands for new books which will follow these developments. New measures require new books, and it is hoped that this new series will not only assist in meeting these requirements, but will incorporate those new methods and processes which have been introduced during the war years.

Fundamentally the books of the series are based, whenever this is necessary, on the Course system, which is now an established feature of technical education. Accordingly, in basic subjects, there will be suitable volumes for the successive years of the course. It is recognised that there must be differences of opinion among teachers as to the sequence and contents of the work in the different years of a Course. But in general it is hoped that the treatment of the subjects will be sufficiently flexible to meet the needs of most teachers and students.

P. ABBOTT.

CONTENTS

CHAPTER 1

INDICES AND LOGARITHMS

1. In a previous course the student will have learnt how to use indices and logarithms in elementary applications. A general treatment will now be given, so that it may be possible to proceed to more difficult examples.

The Index Notation

If a be any number, and n be a positive integer, then a^n means $a \times a \times a \times \ldots$ to n factors, or, *By* a^n *is meant the product of n factors each of which is a ;* a^n *is termed the nth power of a.*

2. Laws of Indices

(1) Multiplication

To prove $a^m \times a^n = a^{m+n}$, where m and n are positive integers.

By definition above,

$$a^m = a \times a \times a \times \ldots \text{ to } m \text{ factors}$$

and $$a^n = a \times a \times a \times \ldots \text{ to } n \text{ factors}$$

$$\therefore \quad a^m \times a^n = (a \times a \times a \times \ldots \text{ to } m \text{ factors}) \times (a \times a \times a \times \ldots \text{ to } n \text{ factors})$$

Thus there are $(m + n)$ factors, each of which is a, on the right-hand side.

$$\therefore \quad a^m \times a^n = a \times a \times a \times \ldots \text{ to } (m + n) \text{ factors,}$$

or $$a^m \times a^n = a^{m+n} \text{ (by definition)}$$

(2) Division

To prove $a^m \div a^n = a^{m-n}$, where m and n are positive integers, and m is greater than n.

By definition,

$$a^m = a \times a \times a \times \ldots \text{ to } m \text{ factors}$$

and

$$a^n = a \times a \times a \times \ldots \text{ to } n \text{ factors}$$

$$\therefore \quad a^m \div a^n = \frac{a \times a \times a \times \ldots \text{ to } m \text{ factors}}{a \times a \times a \times \ldots \text{ to } n \text{ factors}}$$

Cancelling n of the m factors in the numerator by a corresponding number of factors in the denominator, we are left with $(m - n)$ factors in the numerator.

$$\therefore \quad a^m \div a^n = a \times a \times a \times \ldots \text{ to } (m - n) \text{ factors}$$

or

$$a^m \div a^n = a^{m-n} \text{ (by definition)}$$

Note. The case in which n is greater than m will be dealt with later.

(3) Power of a Power

To prove $(a^m)^n = a^{mn}$.

By definition,

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots \text{ to } n \text{ factors}\\ &= a^{m+m+m \ldots \text{ to } n \text{ terms}} \text{ (First Law of Indices)}\\ &= a^{mn}\end{aligned}$$

3. Extension of the Meaning of an Index

The Laws of Indices which we have considered are based on the definition of a power in which the index is a positive integer. If the index is a fraction, or a negative number, or zero, the definition ceases to have any intelligent meaning. But Algebra aims at generalisation, and if Indices are to have a really practical value, we must be able to use them in *all* cases, and not merely subject to the restriction stated above.

It becomes necessary, therefore, to consider what mean-

ings can be attached to powers in which the indices are no longer positive integers.

It is important that we should be clear as to what principle must guide us in thus extending the meaning of a power. The principle, clearly, must be this:—

If the new quantities are to be regarded as Indices, *they must obey the Laws of Indices which have already been formulated.* They must be governed by the same laws as when the Indices are positive integers.

(1) *To find a meaning for* $a^{\frac{1}{n}}$.

Let us first consider a simple case, viz.:

To find a meaning for $a^{\frac{1}{2}}$.

By the principle stated above, $a^{\frac{1}{2}}$ must obey the fundamental Laws of Indices.

$$\begin{aligned} \therefore \quad a^{\frac{1}{2}} \times a^{\frac{1}{2}} &= a^{\frac{1}{2}+\frac{1}{2}} \\ &= a \end{aligned}$$

Thus $a^{\frac{1}{2}}$ must be such a quantity that, on being multiplied by itself, the result is a. But, by definition, such a quantity is the square root of a.

$$\therefore \quad \text{by } a^{\frac{1}{2}} \text{ we mean } \sqrt{a}$$

Similarly, $$a^{\frac{1}{3}} = \sqrt[3]{a}$$

We may now proceed to the general case.

Meaning of $a^{\frac{1}{n}}$.

Since the First Law of Indices must be obeyed,

then $$\begin{aligned} a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{ to } n \text{ factors} \\ &= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}\ldots \text{ to } n \text{ terms}} \\ &= a^1 \\ &= a \end{aligned}$$

Hence $$a^{\frac{1}{n}} = \sqrt[n]{a}$$

(2) *To find a meaning for* $a^{\frac{m}{n}}$.

In a simple case,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} \text{ (by First Law of Indices)}$$
$$= a^2$$

Hence $a^{\frac{2}{3}}$ must be the cube root of a^2,

or $$a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

In general,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots \text{ to } n \text{ factors}$$
$$= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n} \ldots \text{ to } n \text{ terms}} \text{ (by First Law of Indices)}$$
$$= a^{\frac{m}{n} \times n}$$
$$= a^m$$

Hence $a^{\frac{m}{n}}$ must be the nth root of a^m,

or $$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

(3) *To find a meaning for* a^0.

If the First Law of Indices is to hold when the index is zero,

then $$a^0 \times a^n = a^{0+n}$$
$$= a^n$$
$$\therefore \quad a^0 = a^n \div a^n$$
$$= 1$$

This result is independent of the value of a.

$\therefore$ *For all values of* a *we define* a^0 *as* 1.

(4) *To find a meaning for* a^{-n}.

Assuming as before that the Laws of Indices hold for negative indices,

Then by the First Law

$$a^n \times a^{-n} = a^{n-n}$$
$$= a^0$$
$$= 1$$
$$\therefore \quad a^{-n} = 1 \div a^n = \frac{1}{a^n}$$

Thus *a^{-n} is to be defined as the reciprocal of a^n.*

For example,

$$a^{-2} = \frac{1}{a^2}$$

$$a^{-1} = \frac{1}{a}$$

$$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}$$

Similarly $$\frac{1}{a^{-2}} = a^2$$

or generally $$\frac{1}{a^{-n}} = a^n$$

Worked Examples

1. *Find the value of* $2^{\frac{3}{2}}$, given $\sqrt{2} = 1{\cdot}414$.

$$\begin{aligned} 2^{\frac{3}{2}} &= 2^{1+\frac{1}{2}} \\ &= 2^1 \times 2^{\frac{1}{2}} \\ &= 2 \times \sqrt{2} = \mathit{2{\cdot}828} \end{aligned}$$

2. *Find the value of* $2^{-\frac{1}{2}}$.

$$\begin{aligned} 2^{-\frac{1}{2}} &= \frac{1}{2^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \text{ (rationalising)} \\ &= \frac{1{\cdot}414}{2} = \mathit{0{\cdot}707} \end{aligned}$$

3. *Find the value of* $(16^{\frac{1}{4}})^3$.

$$\begin{aligned} (16^{\frac{1}{4}})^3 &= (\sqrt[4]{16})^3 \\ &= 2^3 = \mathit{8} \end{aligned}$$

4. *Find the value of* $(\frac{2}{3})^{-\frac{3}{2}}$.

$$
\begin{aligned}
(\tfrac{2}{3})^{-\frac{3}{2}} &= \frac{1}{(\frac{2}{3})^{\frac{3}{2}}} \\
&= \frac{1}{\dfrac{(2)^{\frac{3}{2}}}{(3)^{\frac{3}{2}}}} \\
&= \frac{(3)^{\frac{3}{2}}}{(2)^{\frac{3}{2}}} \\
&= (\tfrac{3}{2})^{\frac{3}{2}} \\
&= \sqrt{(\tfrac{3}{2})^3} \\
&= \sqrt{\tfrac{27}{8}} \\
&= \sqrt{\tfrac{9}{4} \times \tfrac{3}{2}} \\
&\Rightarrow \tfrac{3}{2}\sqrt{\tfrac{3}{2}} \\
&= 1{\cdot}5\sqrt{1{\cdot}5} \\
&= 1{\cdot}5 \times 1{\cdot}2247 = \mathit{1{\cdot}837}
\end{aligned}
$$

5. *Evaluate* $(25^{0\cdot125})^{-4}$.

$$
\begin{aligned}
(25^{0\cdot125})^{-4} &= \frac{1}{(25^{0\cdot125})^4} \\
&= \frac{1}{25^{0\cdot5}} \\
&= \frac{1}{25^{\frac{1}{2}}} \\
&= \frac{1}{\sqrt{25}} = \frac{\mathit{1}}{\mathit{5}}
\end{aligned}
$$

EXERCISE *1*

1. Write down the values of $8^{\frac{2}{3}}$, $25^{\frac{3}{2}}$, $(16^{\frac{1}{4}})^{\frac{3}{2}}$.
2. Write down the values of 2^{-2}, $(5^{-1})^2$, $\frac{1}{10^{-2}}$, $\frac{2}{2^{-3}}$.
3. Write down the values of $(\frac{1}{2})^{-2}$, $(\frac{2}{3})^{-3}$, $(16)^{0\cdot5}$.
4. Write down the values of $(36)^{-0\cdot5}$, $(4)^{1\cdot5}$, $(\frac{1}{4})^{2\cdot5}$.
5. Find the values of $3^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{3}}$ (to 3 places of decimals).
6. Find the values of $10^{\frac{1}{2}}$, 10^{-2}, 10^{-1}, $10^{-0\cdot5}$.

7. Find the values of $2^{\frac{1}{2}}$, $2^{\frac{1}{3}}$ (to 2 places of decimals).

8. Find the values of $(16^{-0.25})^3$, $(8^{0.5})^{-3}$.

9. Find the simplest form of $a^4 \times a^{-2} \times a^{\frac{1}{2}}$.

10. Find the simplest form of $a^{\frac{1}{2}} \times a^{\frac{1}{3}} \times a^{-1}$.

11. If $10^{\frac{1}{3}} = 2{\cdot}154$, find the value of $10^{\frac{2}{3}}$ (to 2 places of decimals).

12. Write down the square root of $81a^4b^2$ and the cube root of $8a^6b^3$.

Logarithms

4. The student has learnt in his previous work that it is possible to express any number as a power of any other number. In particular, he has learnt that any number can be expressed as a power of 10. By the extension of the meaning of an index to include fractions, negative numbers and zero, it is now possible to include cases not previously considered. The following table shows a few such cases, all of which can be determined by the student himself by using the previous exercises.

Power . .	10^{-2}	10^{-1}	$10^{-0.5}$	10^0	$10^{0.5}$	10^1	$10^{1.5}$	10^2
Number . .	0·01	0·1	0·3162	1	3·162	10	31·62	100
Index . .	−2	−1	−0·5	0	0·5	1	1·5	2

This table could be extended indefinitely, both for positive and negative indices. It will be seen that it is possible to obtain a series of indices extending from $-\infty$, through zero, to $+\infty$, which will correspond to a series of numbers, extending from zero to infinity, expressed as powers of 10.

It is also possible to take any other number, say a, and in a similar way to obtain a system of indices corresponding to numbers expressed as powers of a.

Such a system of indices, by means of which a series of numbers is expressed as powers of another number, called the *base* of the system, is called *a system of logarithms*.

Hence we may define a logarithm as follows:

A logarithm of a number to a given base is the Index which indicates what power the number is of the base.

For example, since

$$1000 = 10^3$$

the above definition enables us to say that

3 is the logarithm of 1000 to base 10

which may be written

$$3 = \log_{10} 1000$$

Similarly, since

$$0{\cdot}01 = 10^{-2}$$

-2 is the logarithm of 0·01 to base 10, which may be written

$$-2 = \log_{10} 0{\cdot}01$$

In general,

if $$y = a^x$$

then x is the index which indicates what power y *is of* a; so, by definition,

x is the logarithm of y to base a

or $$x = \log_a y$$

It should be noted that the two equations,

$$y = a^x$$

and $$x = \log_a y$$

both express the relations which, as shown above, exist between the three numbers, x, y and a. They are different forms expressing the same thing, and the student should be able readily to change from one to the other.

Laws of Logarithms

5. The student has previously learnt that logarithms are of great practical value in enabling us to carry out operations with numbers. He has also learnt rules for these

operations. We will now proceed to examine these rules again, and to consider proofs of them.

The fundamental principle which will guide our work is that *a Logarithm is an Index,* and therefore *the Laws of Logarithms must correspond to the Laws of Indices.*

(1) Logarithm of a Product

Let x and y be numbers, such that

$$x = a^m \quad \text{or} \quad m = \log_a x$$

and

$$y = a^n \quad \text{or} \quad n = \log_a y$$

Then
$$\begin{aligned} x \times y &= a^m \times a^n \\ &= a^{(m+n)} \text{ (First Law of Indices)} \\ \therefore \quad \log_a (x \times y) &= m + n \\ &= \log_a x + \log_a y \end{aligned}$$

$\therefore$ *The logarithm of the product of two numbers is the sum of their logarithms.*

The student should compare this rule with the First Law of Indices on p. 11 (remembering that logarithms are indices). This may be extended to any number of factors.

Thus $\log_a (x \times y \times z) = \log_a x + \log_a y + \log_a z.$

(2) Logarithm of a Quotient

Let x and y be numbers such that

$$x = a^m \quad \text{or} \quad m = \log_a x$$

and

$$y = a^n \quad \text{or} \quad n = \log_a y$$

Then
$$\begin{aligned} x \div y &= a^m \div a^n \\ &= a^{(m-n)} \text{ (Second Law of Indices)} \\ \therefore \quad \log_a (x \div y) &= m - n \\ &= \log_a x - \log_a y \end{aligned}$$

$\therefore$ *The logarithm of a quotient is obtained by subtracting the logarithm of the divisor from the logarithm of the dividend.*

(3) Logarithm of a Power

Let $$x = a^n \text{ or } n = \log_a x$$
Then $$x^m = (a^n)^m$$
$$= a^{mn} \text{ (Third Law of Indices)}$$
$$\therefore \quad \log_a x^m = m \times n$$
$$= m \times \log_a x$$

$\therefore$ *The logarithm of any power of a number is equal to the product of the logarithm of the number and the index of the power.*

Since, as stated on p. 13, the Laws of Indices must apply to all indices, the above Law will hold for all values of m.

Thus if
$$m = \frac{1}{p}$$
Then
$$\log_a x^{\frac{1}{p}} = \frac{1}{p} \times \log_a x$$
and, since
$$x^{\frac{1}{p}} \text{ means } \sqrt[p]{x} \text{ (see p. 13)}$$
$$\therefore \quad \log \sqrt[p]{x} = \frac{1}{p} \log_a x$$

Similarly for a negative index:

If $$m = -p$$
then $$\log_a (x)^{-p} = -p \log_a x$$

6. Some Properties of Logarithms

We now summarise these laws, together with some special cases:

1. *The logarithm of the base itself is always unity.*

Since $a^1 = a \quad \therefore \quad \log_a a = 1$

2. *The logarithm of 1 is always zero, whatever the base.*

Since $a^0 = 1 \quad \therefore \quad \log_a 1 = 0$

3. *The logarithm of a number is equal to minus the logarithm of its reciprocal.*

$$\log_a x = -\log_a \left(\frac{1}{x}\right)$$

4. $$\log_a (x \times y) = \log_a x + \log_a y$$
5. $$\log_a (x \div y) = \log_a x - \log_a y$$
6. $$\log_a (x)^n = n \log_a x$$
7. $$\log_a \sqrt{x} = \frac{1}{n} \log_a x$$
8. $$\log_a (x)^{-n} = -n \log_a x$$

7. Application of Laws of Logarithms to Calculations

We will now proceed to apply these rules to a few calculations:

1. *Evaluate*

$$\sqrt{\frac{0{\cdot}0972 \times 19{\cdot}98}{6{\cdot}38 \times 0{\cdot}00939}}$$

Let $$x = \sqrt{\frac{0{\cdot}0972 \times 19{\cdot}98}{6{\cdot}38 \times 0{\cdot}00939}}$$

Taking logs,

$$\begin{aligned} \log x &= \tfrac{1}{2}\{\log 0{\cdot}0972 + \log 19{\cdot}98 - (\log 6{\cdot}38 + \log 0{\cdot}00939)\} \\ &= \tfrac{1}{2}\{\bar{2}{\cdot}9877 + 1{\cdot}3007 - (0{\cdot}8048 + \bar{3}{\cdot}9727)\} \\ &= \tfrac{1}{2}\{0{\cdot}2884 - \bar{2}{\cdot}7775\} \\ &= \tfrac{1}{2}\{1{\cdot}5109\} \\ &= 0{\cdot}7555 \\ &= \log (5{\cdot}696) \end{aligned}$$

$\therefore\ x = 5{\cdot}696$

Note.—In performing the step $0{\cdot}2884 - \bar{2}{\cdot}7775$, first write $0{\cdot}2884$ as $\bar{1} + 1{\cdot}2884$.

Then from $\bar{1} + 1{\cdot}2884$
take $\bar{2} + 0{\cdot}7775$

$1 + 0{\cdot}5109 = 1{\cdot}5109$

2. *Evaluate*

$$\sqrt[3]{0{\cdot}0838}$$

Let $$x = \sqrt[3]{0{\cdot}0838}$$

Taking logs,

$$\begin{aligned} \log x &= \tfrac{1}{3} \log 0{\cdot}0838 \\ &= \tfrac{1}{3}\{\bar{2}{\cdot}9232\} \\ &= \tfrac{1}{3}\{\bar{3} + 1{\cdot}9232\} \\ &= \bar{1} + 0{\cdot}6411 \\ &= \bar{1}{\cdot}6411 \\ &= \log (0{\cdot}4376) \\ \therefore \quad x &= \mathit{0{\cdot}4376} \end{aligned}$$

Note.—It should be remembered that the mantissa (the decimal part) of a logarithm must always be positive, or the log tables cannot be used for it. We must, therefore, adjust $\bar{2}{\cdot}9232$, so that we can divide it by 3, and still keep a positive mantissa. This is done as shown. By writing $\bar{2}$ as $\bar{3} + 1$, we can get an *exact* division of the negative characteristic, leaving $+ 1$, which can be carried on to the mantissa as it also is positive. Study the following which might arise in the process of taking a root:

$\dfrac{\bar{1}{\cdot}6235}{2}$ would be written $\dfrac{\bar{2} + 1{\cdot}6235}{2} = \bar{1}{\cdot}8118$

$\dfrac{\bar{3}{\cdot}6235}{2}$ „ „ „ $\dfrac{\bar{4} + 1{\cdot}6235}{2} = \bar{2}{\cdot}8118$

$\dfrac{\bar{4}{\cdot}6235}{3}$ „ „ „ $\dfrac{\bar{6} + 2{\cdot}6235}{3} = \bar{2}{\cdot}8745$

3. *Evaluate*

$$(0{\cdot}0273)^{\frac{2}{3}}$$

Let $$x = (0{\cdot}0273)^{\frac{2}{3}}$$

Taking logs,

$$\begin{aligned} \log x &= \tfrac{2}{3} \log (0{\cdot}0273) \\ &= \tfrac{2}{3} \times (\bar{2}{\cdot}4362) \\ &= \frac{\bar{4}{\cdot}8724}{3} \\ &= \frac{\bar{6} + 2{\cdot}8724}{3} \\ &= \bar{2}{\cdot}9575 \\ &= \log 0{\cdot}09067 \\ \therefore \quad x &= \mathit{0{\cdot}09067} \end{aligned}$$

4. *Find the value of*

$$(6{\cdot}023)^{-2{\cdot}5}$$

Let $$x = (6{\cdot}023)^{-2{\cdot}5}$$

Taking logs,

$$\begin{aligned}\log x &= -2{\cdot}5 \times \log 6{\cdot}023\\ &= -2{\cdot}5 \times 0{\cdot}7798\\ &= -1{\cdot}9495\\ &= \bar{2} + 0{\cdot}0505\\ &= \bar{2}{\cdot}0505\\ &= \log 0{\cdot}01123\\ \therefore\quad x &= 0{\cdot}01123\end{aligned}$$

Note.—In this example, it is seen that in multiplying 0·7798 by $-$ 2·5, we obtain $-$ 1·9495 which is wholly negative. Before using the log tables, we adjust this as $-2 + 0{\cdot}0505$, thus separating it into negative characteristic and positive mantissa, which can then be used in the usual way.

(Alternatively, we could write $(6{\cdot}023)^{-2{\cdot}5}$ as $\frac{1}{(6{\cdot}023)^{2{\cdot}5}}$, etc.)

5. *Evaluate*

$$(0{\cdot}1276)^{-1{\cdot}7}$$

Let $$x = (0{\cdot}1276)^{-1{\cdot}7}$$

Taking logs,

$$\begin{aligned}\log x &= -1{\cdot}7 \times \log 0{\cdot}1276\\ &= -1{\cdot}7 \times \bar{1}{\cdot}1059\\ &= (-1{\cdot}7 \times -1) + (0{\cdot}1059 \times -1{\cdot}7)\\ &= 1{\cdot}7 - 0{\cdot}18003\\ &= 1{\cdot}51997\\ &= \log 33{\cdot}11\\ \therefore\quad x &= 33{\cdot}11\end{aligned}$$

6. *Evaluate*

$$(0{\cdot}0719)^{2{\cdot}4}$$

Let $$x = (0{\cdot}0719)^{2{\cdot}4}$$

Taking logs,

$$\begin{aligned}
\log x &= 2{\cdot}4 \log 0{\cdot}0719 \\
&= 2{\cdot}4 \times \bar{2}{\cdot}8567 \\
&= (2{\cdot}4 \times -2) + (2{\cdot}4 \times 0{\cdot}8567) \\
&= -4{\cdot}8 + 2{\cdot}05608 \\
&= 2{\cdot}05608 - 4{\cdot}8 \\
&= \bar{3}{\cdot}25608 \\
&= \log 0{\cdot}001803 \\
\therefore \quad x &= \mathit{0{\cdot}001803}
\end{aligned}$$

Note.—The subtraction $(2{\cdot}05608 - 4{\cdot}8)$ is done as it is usually done in logarithms:

From	2·05608
Take	4·8
	$\bar{3}{\cdot}25608$

thus leaving only the characteristic negative.

8. Solution of Exponential Equations

When the unknown quantity is in the index of a term (or terms) of an equation, we can solve the equation, called an exponential equation, by using logarithms.

1. *Solve for x*

$$4^x = (9{\cdot}1)^{x-2}$$

Taking logs of both sides,

$$\begin{aligned}
x \log 4 &= (x-2)(\log 9{\cdot}1) \\
\therefore \quad 0{\cdot}6021x &= 0{\cdot}9590(x-2) \\
\therefore \quad 0{\cdot}6021x &= 0{\cdot}9590x - 1{\cdot}918 \\
\therefore \quad 0{\cdot}3569x &= 1{\cdot}918 \\
\therefore \quad x &= \frac{1{\cdot}918}{0{\cdot}3569} \\
\therefore \quad x &= \mathit{5{\cdot}374}
\end{aligned}$$

Note.—The student must be careful *to divide 1·918 by 0·3569*, either by arithmetic, or by subtracting the log of 0·3569 from the log of 1·918 in the usual way: it is incorrect to subtract 0·3569 from 1·918 on the assumption that these numbers are logarithms.

2. *Solve for x*

$$2^x \cdot 5^{x-3} = 7^{1-x}$$

Taking logs of both sides,

$$x \log 2 + (x - 3) \log 5 = (1 - x) \log 7$$
$$\therefore \quad 0{\cdot}3010x + 0{\cdot}6990\,(x - 3) = 0{\cdot}8451(1 - x)$$
$$\therefore \quad 0{\cdot}3010x + 0{\cdot}6990x - 2{\cdot}097 = 0{\cdot}8451 - 0{\cdot}8451x$$
$$\therefore \quad 1{\cdot}8451x = 2{\cdot}9421$$
$$\therefore \quad x = 1{\cdot}594$$

EXERCISE 2

Evaluate using logarithms :

1. $\sqrt[5]{0{\cdot}0263}$.
2. $(92{\cdot}01)^{-1{\cdot}3}$.
3. $(16{\cdot}7)^{-0{\cdot}9}$.
4. $2 \times 19{\cdot}02^{-0{\cdot}3}$.
5. $(0{\cdot}00882)^{-1{\cdot}7}$.
6. $(0{\cdot}00259)^{0{\cdot}18}$.
7. $(0{\cdot}017)^{0{\cdot}017}$.
8. $(0{\cdot}25)^{0{\cdot}25}$.
9. $\dfrac{2}{(0{\cdot}0619)^{0{\cdot}03}}$.
10. $(0{\cdot}626)^{0{\cdot}5} \times (9{\cdot}002)^{-0{\cdot}62}$.
11. $(13{\cdot}27)^{0{\cdot}6} \times \log 2{\cdot}718$.
12. $(0{\cdot}9929)^{\frac{1}{3}}$.
13. $\dfrac{92{\cdot}26^{0{\cdot}7}}{\log 2{\cdot}718}$.
14. $(0{\cdot}213)^{\frac{1}{2}} \times (7{\cdot}16)^{-\frac{1}{3}}$.
15. $\dfrac{1{\cdot}521^{-0{\cdot}2}}{\sqrt[6]{1{\cdot}923}}$.
16. ae^{-kt} when $a = 6$, $e = 2{\cdot}718$, $k = 45$ and $t = 0{\cdot}0037$.

Solve for x in Nos. 17 and 18 :

17. $x^{3{\cdot}6} = 200$.
18. $500x^{-2{\cdot}8} = 5{\cdot}16$.
19. Solve for y : $7{\cdot}16^y = 1{\cdot}92^{y+2}$.
20. Solve for p : $3^p \cdot 5^{p-2} = 8^{2p} \cdot 7^{1-p}$.
21. In the equation, $n = K \cdot H^{1{\cdot}25} \cdot P^{-0{\cdot}5}$, find K, when H = 12, $n = 200$, and P = 80.
22. A gas is expanding according to the law $pv^n = C$.
    (*a*) Find C when $p = 92$, $v = 2{\cdot}6$ and $n = 1{\cdot}3$.
    (*b*) Find v when C = 310, $p = 96$ and $n = 1{\cdot}3$.
    (*c*) Find n when C = 330, $p = {\cdot}91$ and $v = 2{\cdot}5$.

23. If $C = 2{\cdot}6(1000a)^{0{\cdot}82}$, find a when $C = 15$.

24. Given that $A = P\left(1 + \frac{r}{100}\right)^n$, find n when $A = 303{\cdot}9$, $P = 250$ and $r = 5$.

25. In the equation $R = 2{\cdot}9 + \frac{V^4}{0{\cdot}03L + 61}$, find V, when $R = 12{\cdot}96$ and $L = 80$.

26. Given $v = \frac{C\sqrt{2gh}}{\sqrt{1 + \left(\frac{1}{K} - 1\right)^2}}$, find v when $C = 0{\cdot}82$, $K = 0{\cdot}61$, $h = 60$ and $g = 32{\cdot}2$.

27. If $C = C_0 e^{-\frac{Rt}{L}}$, find C when $C_0 = 25$, $R = 0{\cdot}28$ $t = 0{\cdot}015$, $L = 0{\cdot}003$ and $e = 2{\cdot}718$.

28. In the equation of No. 27, find t if $C_0 = 25$, $L = 0{\cdot}003$, $C = 1{\cdot}6$, $e = 2{\cdot}718$ and $R = 0{\cdot}28$.

29. If $C = \frac{V}{R}(1 - e^{-\frac{Rt}{L}})$, find L when $C = 15$, $V = 240$, $R = 10$, $t = 0{\cdot}03$ and $e = 2{\cdot}718$.

30. If $\frac{T}{t} = e^{\mu r\theta}$, find θ when $T = 65$, $t = 42$, $r = 1{\cdot}8$, $e = 2{\cdot}718$ and $\mu = 0{\cdot}2$.

31. In a spring-loaded governor the number N (revolutions per minute) occurs.

$$p = 2\left(\frac{WRN^2y}{35240Z} - \frac{W}{Z}\right) - V$$

Change round the formula so as to express N in terms of the other quantities.

Then using logarithms as much as possible, calculate N when $p = 260$, $W = 11{\cdot}5$, $R = 5{\cdot}2$, $y = 4{\cdot}9$, $Z = 3{\cdot}7$, $V = 14$. (U.E.I., 1935.)

32. The velocity, V, of water in a pipe of diameter d *feet* is given by the expression

$$V = 4{\cdot}01 \times \sqrt{\frac{duH}{fL}}$$

Express d in terms of the other quantities.

Then calculate by means of logarithms the diameter of the pipe in *inches* if $V = 36$, $u = \frac{1}{3}$, $H = 1725$, $f = 0{\cdot}007$, $L = 500$. (U.E.I., 1932.)

33. (i) Express each of the three numbers $[1000^{\frac{1}{3}}]^{\frac{1}{4}}$, $0{\cdot}83$, $\dfrac{1}{0{\cdot}38}$ as a power of 10.

(ii) For what value of x is $5^x = 50$?

(iii) Calculate the value of $\sqrt[3]{HK^2 + K^3}$ when $H = 12{\cdot}9$, $K = 7{\cdot}32$. (N.C.T.E.C., 1934.)

34. (i) Express each of the numbers $0{\cdot}3802$ and $\dfrac{1}{\sqrt[3]{0{\cdot}3802}}$ as a power of 10.

(ii) For what value of x is $50^x = 100$?

(iii) Calculate the value of $\dfrac{4{\cdot}7(P + Q)}{P\sqrt{P^2 - Q^2}}$ when $P = 18{\cdot}35$, $Q = 14{\cdot}85$. (N.C.T.E.C., 1935.)

35. In a belt drive the ratio of the tension T_1 on the tight side of the belt to the tension T_2 on the slack side is given by the equation

$$\frac{T_1}{T_2} = e^{\mu a},$$

where $e = 2{\cdot}718$, "u" is the coefficient of friction between the belt and the pulley and "a" the angle of lap of the belt in radians, whilst the effective pull of the belt is $T_1 - T_2$. If $u = 0{\cdot}27$, $a = 165$ degrees and Effective Pull $= 160$ lb., find T_1 and T_2. Without actually working out, state the effect on the ratio of the tensions of doubling the value of "a." (U.E.I., 1932.)

CHAPTER 2

LOGARITHMS TO OTHER BASES THAN 10

1. Relation between the Logarithms of a Number to Different Bases

Hitherto in using logarithms the student has used only those which are calculated to base 10, although it should be noted that the laws which govern the use of logarithms have been shown to be true whatever the base.

But the student will find that the logarithms which he has to use, especially in Engineering, are not always referred to 10 as a base. Consequently it is necessary to consider in these cases what relation such logarithms bear to those calculated to base 10, which he finds in his tables.

In simple cases, the logarithms to other bases can easily be determined.

For example, since $5^2 = 25$, the logarithm of 25 to base 5 is 2, or, with the usual notation,

$$\log_5 25 = 2$$

Similarly $$\log_4 64 = 3$$

The student will note that it is very important to specify the base. In actual computations, however, the logarithm tables are calculated to base 10, and, in working, this base is usually omitted as being understood.

2. To Obtain Logarithms to Other Bases than 10

In cases such as we have considered above, where the index or logarithm is a positive integer, we can usually obtain this by inspection.

For example, if we require $\log_6 216$.

By definition, if x is the required logarithm, then we must have $6^x = 216$.

Hence we can see that $x = 3$, since $6^3 = 216$, and

$$\log_6 216 = 3$$

Now consider a more difficult case.

To find $\log_3 10$.

If x be the required logarithm, then, by definition,

$$3^x = 10$$

Taking logs. to base 10,

$$x \log_{10} 3 = \log_{10} 10$$
$$\therefore \quad x \log_{10} 3 = 1$$
$$\therefore \quad x = \frac{1}{\log_{10} 3} = \frac{1}{0{\cdot}4771}$$
$$\therefore \quad \log_3 10 \doteq 2{\cdot}096$$

To evaluate $\log_3 100$.

We could proceed as above, or, employing the Third Law of Indices, we could use the following method.

Since $\quad 100 = 10^2 \qquad \therefore \quad 2 = \log_{10} 100$

But, as found above, $10 = 3^{2\cdot096}$.

Then $\quad 100 = 10^2 = (3^{2\cdot096})^2$

$$\therefore \quad \log_3 100 = \log_3 (3^{2\cdot096})^2$$
$$= 2 \times 2{\cdot}096$$
$$= \log_{10} 100 \times \log_3 10$$

We now proceed to give a general treatment of this.

To Find the Relation between the Logarithms of a Number to Different Bases

Let N be any number and a and b two bases.

Let $\quad N = b^y \quad$ and $\qquad \therefore y = \log_b N$

$$\therefore \quad \log_a N = \log_a (b^y)$$
$$= y \log_a b$$
$$= \log_b N \times \log_a b$$

This important result should be carefully remembered.

$$\boldsymbol{\log_a N = \log_b N \times \log_a b}$$

It will be noticed that the factor $\log_a b$ is always the same whatever the value of N.

$\therefore$ if we know the log. of N to a base b, and we require the log. to base a, we multiply the known log. of N to the base b, by the log. of the base b to the new base a. This constant multiplier is termed a *Modulus*.

Note.

Since $\log_a N = \log_b N \times \log_a b$ for all values of N, let $N = a$.

Then $\qquad \log_a a = \log_b a \times \log_a b$

But $\qquad \log_a a = 1$

$$\therefore \quad \log_b a \times \log_a b = 1$$

and $$\log_a b = \frac{1}{\log_b a}$$

which is a useful result.

It might help the student to remember the formula

$$\log_a N = \log_b N \times \log_a b$$

by noticing the sequence of the letters in

$$\frac{N}{a} = \frac{N}{\cancel{b}} \times \frac{\cancel{b}}{a}$$

Napierian or Natural Logarithms

3. Logarithms were first discovered by Lord Napier in the sixteenth century. As he used them, they were not calculated to base 10, but to a base denoted by the letter "e," where

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots \textit{ ad}$$

infinitum.

In this series, the terms are continually diminishing and ultimately become indefinitely small. To calculate the value of e to any required degree of accuracy, we take as many terms as may be required for the purpose. Its value correct to 5 places of decimals is 2·71828.

To the student who has not progressed very far in mathematics, the choice of such a base is difficult to understand; but when his studies are sufficiently advanced, he will discover that logarithms calculated to this base enter naturally into the higher branches of mathematics, and indeed are used almost exclusively except when computations are necessary. Hence the term "natural logarithms." They are also termed "Hyperbolic logarithms." The student will also find that these logarithms enter largely into his work in Engineering and Physics, and consequently a knowledge of their properties is essential.

For numerical computations, the Napierian logarithms are inconvenient, and it is much more advantageous to use the common logarithms, since the base 10 to which they are calculated is also the base of the common scale of notation. One advantage is that numbers with the same significant figures have the same mantissa or decimal part, the characteristic or integral part varying with the position of the decimal point in the number. With this the student is already acquainted, and can work out the proof for himself.

It was Henry Briggs, a Professor of Mathematics at Oxford, who saw the great advantage of common logarithms in calculations, and it was he who first compiled a table of logarithms calculated to base 10.

4. To Convert Napierian Logarithms to Common Logarithms

The student will frequently need, in the course of his work, to find the numerical value of the Napierian logs. of numbers, since tables of these are not usually accessible. In order to do this, he must know the relation between these and the common logarithms of his tables. The conversion is effected by using the formula on p. 29, viz.:

$$\log_a N = \log_b N \times \log_a b$$

Adapting this, we have :

$$\log_e N = \log_{10} N \times \log_e 10$$

Also using the result on p. 30, we have :

$$\log_e 10 = \frac{1}{\log_{10} e}$$

Consequently to make the change we use either,

$$\mathbf{\log_e N = \log_{10} N \times \log_e 10}$$

or
$$\mathbf{\log_e N = \log_{10} N \times \frac{1}{\log_{10} e}}$$

Now from the tables,

$$\log_{10} e = 0{\cdot}4343$$

$$\therefore \quad \log_e 10 = \frac{1}{0{\cdot}4343} = 2{\cdot}3026$$

Hence

$$\mathbf{\log_e N = \log_{10} N \times 2{\cdot}3026}$$

or
$$\mathbf{\log_e N = \log_{10} N \div 0{\cdot}4343}$$

Examples

1. *To find $\log_e 50$.*

From tables $\log_{10} 50 = 1{\cdot}6990$

$$\therefore \quad \log_e 50 = 1{\cdot}6990 \times 2{\cdot}3026$$

Let
$$x = 1{\cdot}6990 \times 2{\cdot}3026$$

then
$$\begin{aligned} \log x &= \log 1{\cdot}6990 + \log 2{\cdot}3026 \\ &= 0{\cdot}2303 + 0{\cdot}3622 \\ &= 0{\cdot}5925 \\ &= \log 3{\cdot}913 \\ \therefore \quad x &= 3{\cdot}913 \end{aligned}$$

or
$$\log_e 50 = 3{\cdot}913$$

Note.—**As $\log_{10} 2{\cdot}3026$ will constantly be used in such calculations, the student should note that $\log_{10} 2{\cdot}3026 = 0{\cdot}3622$.**

2. *Find the value of Q from the formula*

$$Q = \log_e \frac{T}{461} + \frac{1420}{T} - 0{\cdot}65$$

when $\quad T = 600$

$$\begin{aligned}\log_e \tfrac{600}{461} &= \log_{10} \{\tfrac{600}{461}\} \times 2{\cdot}3026\\ &= \{\log10_{10} 600 - \log_{10} 461\} \times 2{\cdot}3026\\ &= \{2{\cdot}7782 - 2{\cdot}6637\} \times 2{\cdot}3026\\ &= 0{\cdot}1145 \times 2{\cdot}3026\\ &= 0{\cdot}2637\end{aligned}$$

$$\frac{1420}{T} = \tfrac{1420}{600} = 2{\cdot}3667$$

$$\begin{aligned}\therefore \quad Q &= 0{\cdot}2637 + 2{\cdot}3667 - 0{\cdot}65\\ &= 2{\cdot}6304 - 0{\cdot}65\\ &= \mathit{1{\cdot}9804}\end{aligned}$$

3. *Express as simply as possible :—*

(*a*) $b^{\log_b 3}$. (*b*) $b^{2\log_b 3}$.

(N.C.T.E.C., 1935.)

(*a*) Let $\quad x = b^{\log_b 3}$

Taking logs. to base b

$$\begin{aligned}\log_b x &= \log_b 3\\ \therefore \quad x &= 3\end{aligned}$$

(*b*) Let $\quad x = b^{2\log_b 3}$

Taking logs. to base b

$$\begin{aligned}\log_b x &= 2 \log_b 3\\ \therefore \quad \log x &= \log (3)^2\\ \therefore \quad x &= 9\end{aligned}$$

EXERCISE 3

1. Write down, by inspection, the value of :

(*a*) $\log_4 64$. (*c*) $\log_2 64$. (*e*) $\log_{16} 1024$.
(*b*) $\log_5 125$. (*d*) $\log_9 27$. (*f*) $\log_{81} 9$.

2. (*a*) What power of 5 is 50? Find the value of $\log_5 50$.
(*b*) Find $\log_{6\cdot5} 100$.

3. Calculate the value of:

(*a*) $\log_e 4\cdot6$. (*c*) $\log_e 9\cdot6$. (*e*) $\log_e 56$.
(*b*) $\log_e 7\cdot5$. (*d*) $\log_e 40$. (*f*) $\log_e 0\cdot062$.

4. Show that

(*a*) $\log_e 40 = \log_e 8 + \log_e 5$.
(*b*) $\log_e 9\cdot6 = \log_e 2\cdot4 + \log_e 4$.

5. Find the numbers whose logs. to base e are:

(*a*) $1\cdot39$. (*b*) $1\cdot86$. (*c*) $2\cdot205$.
(*d*) $\bar{1}\cdot1560$.

6. Given a table of ordinary logarithms, show how you would find the logarithm of any number N to any base b. Hence calculate $\log_e 2\cdot5$, where $e = 2\cdot718$.
(U.E.I., 1935.)

7. Find the value of $\frac{890}{781} + \log_e \frac{781}{493}$. (U.L.C.I., 1936.)

8. Find the value of $\log_e \dfrac{x}{y}$ when $x = 750$, $y = 682$.
(U.L.C.I., 1928.)

9. (*a*) Prove the relation $\log_e \mathrm{N} = 2\cdot3026 \log_{10} \mathrm{N}$.
(*b*) Find the value of $\log_e 0\cdot6$. (U.L.C.I., 1935.)

10. What is the number whose Napierian logarithm is $1\cdot60952$? (U.L.C.I., 1927.)

11. Given that $\mathrm{H} = 1430 \log_e \dfrac{\mathrm{T}_1}{\mathrm{T}_2} - (\mathrm{T}_1 - \mathrm{T}_2)$, find H when $\mathrm{T}_1 = 623$, and $\mathrm{T}_2 = 320$.

12. Evaluate $\log_e \dfrac{t}{274} + \dfrac{797}{t}$ when $t = 360$.

13. Write in as simple a form as possible

(*a*) $c^{\log_c 5}$. (*c*) $3x^{\log_x 4}$. (*e*) $a^{\frac{1}{\log_x a}}$.

14. The insulation resistance, R megohms, of length l ins., of a certain wire is given by :

$$R = \frac{0{\cdot}42S}{l} \times \log_e \frac{d_2}{d_1}$$

where d_1 and d_2 are the inside and outside diameters of the insulating material and S is the specific resistance in megohms. Find l to the nearest foot when $S = 2000$ megohms, $R = 0{\cdot}44$ megohm, $d_2 = 0{\cdot}3$ cm. and $d_1 = 0{\cdot}16$ cm.

15. Show that

(a) $\log_b a \,.\, \log_c b \,.\, \log_a c = 1.$

(b) $\log_a A \,.\, \log_b B = \log_b A \,.\, \log_a B.$

16. In a calculation on the dryness of steam, the following formula was used :

$$\frac{qL}{T} = \frac{q_1L_1}{T_1} + \log_e \frac{T_1}{T}$$

Use it to find q when $L_1 = 850$, $L = 1000$, $T_1 = 780$, $T = 650$ and $q_1 = 1$.

17. If $m = \dfrac{P(1 + \log_e r)}{r} - B$, find m when $P = 120$, $r = 2{\cdot}5$ and $B = 16$.

18. If $w = 144\{p_1(1 + \log_e r) - r(p_3 + 10)\}$, find w when $p_1 = 110$, $p_3 = 15$ and $r = 3{\cdot}2$.

19. Find T in the equation (used for the entropy of water), $Q = \log_e \dfrac{T}{460}$, when $Q = 0{\cdot}4$.

20. Given $pv^{1{\cdot}3} = C$, find the value of C when $p = 14{\cdot}7$ and $v = 2{\cdot}15$. (U.L.C.I., 1936.)

21. (a) Prove that $\log_b a = \dfrac{\log_c a}{\log_c b}$.

Hence find $\log_e 7$, where $e = 2{\cdot}718$

(b) By expressing each member as a power of 10, find the value of $6{\cdot}92 \times 12{\cdot}46$. (U.E.I.)

CHAPTER 3

QUADRATIC EQUATIONS

1. Equations in which the unknown quantity is in the second, but no higher degree, are called Quadratic Equations, or equations of the second degree.

Thus
$$6x^2 = 54$$
$$5x^2 + 6 = 2x$$
$$7x^2 = 3x - 9$$
are quadratic equations.

In its general form, the quadratic equation can be written:
$$ax^2 + bx + c = 0$$
and all quadratics can be reduced to this form.

2. Methods of Solution

First Method—by Factors

As this method was dealt with in the First-Year Cou it will suffice to consider an example.

Solve: $6x^2 + 11x = 10$

Re-arranging: $6x^2 + 11x - 10 = 0$

$$\therefore \quad (3x - 2)(2x + 5) = 0$$
$$\therefore \quad 3x - 2 = 0 \quad \text{or} \quad 2x + 5 = 0$$
$$\therefore \quad x = \tfrac{2}{3} \quad \text{or} \quad -\tfrac{5}{2}$$

Note.—The right-hand side must be zero before factorising.

Second Method—by Completing the Square

Solve: $5x^2 - 7x - 6 = 0$

Dividing through by 5, the coefficient of x^2,
$$x^2 - \tfrac{7}{5}x - \tfrac{6}{5} = 0$$

Re-arranging:

$$x^2 - \tfrac{7}{5}x = \tfrac{6}{5}$$

Completing the square by adding the square of half the coefficient of x to both sides:

$$x^2 - \tfrac{7}{5}x + (\tfrac{7}{10})^2 = \tfrac{6}{5} + (\tfrac{7}{10})^2$$
$$\therefore \quad x^2 - \tfrac{7}{5}x + (\tfrac{7}{10})^2 = \tfrac{6}{5} + \tfrac{49}{100}$$
$$\therefore \quad x^2 - \tfrac{7}{5}x + (\tfrac{7}{10})^2 = \tfrac{169}{100}$$

Writing left-hand side as a square:

$$(x - \tfrac{7}{10})^2 = \tfrac{169}{100}$$

Taking square root of both sides:

$$x - \tfrac{7}{10} = \pm \tfrac{13}{10}$$
$$\therefore \quad x = \tfrac{7}{10} \pm \tfrac{13}{10}$$
$$\therefore \quad x = \tfrac{20}{10} \quad \text{or} \quad -\tfrac{6}{10}$$
$$\therefore \quad x = 2 \quad \textit{or} \quad -0{\cdot}6$$

Third Method—by Use of Formula

The general form of the quadratic equation

$$ax^2 + bx + c = 0$$

can be solved by using the previous method of completing the square.

Write $ax^2 + bx + c = 0$ as $ax^2 + bx = -c$.

Divide through by a,

$$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Complete the square as before:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Writing as a square

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking the square root of both sides :

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad \boldsymbol{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}}$$

$$\text{or } \boldsymbol{x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

Examples.

1. *To solve* $6x^2 + 11x - 10 = 0$ *by this method note that* $a = 6$, $b = 11$ *and* $c = -10$.

$\therefore$ By substitution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{becomes}$$

$$x = \frac{-11 \pm \sqrt{11^2 - (4)(6)(-10)}}{12}$$

$$\therefore \quad x = \frac{-11 \pm \sqrt{121 + 240}}{12}$$

$$\therefore \quad x = \frac{-11 \pm \sqrt{361}}{12}$$

$$\therefore \quad x = \frac{-11 \pm 19}{12}$$

$$\therefore \quad x = +\tfrac{8}{12} \text{ or } -\tfrac{30}{12}$$

$$\therefore \quad x = \tfrac{2}{3} \text{ or } -2\tfrac{1}{2}$$

2. *Solve:* $5x^2 - 13x + 1 = 0$

$$x = \frac{13 \pm \sqrt{169 - 20}}{10}$$

$$\therefore \quad x = \frac{13 \pm \sqrt{149}}{10}$$

$$\therefore \quad x = \frac{13 \pm 12{\cdot}21}{10}$$

$$\therefore \quad x = \frac{0{\cdot}79}{10} \text{ or } \frac{25{\cdot}21}{10}$$

$$\therefore \quad x = 0{\cdot}079 \text{ or } 2{\cdot}521$$

3. Quadratics with Imaginary Roots

In some cases it will be found that $4ac$ is greater than b^2, in which case we are faced with the problem of finding the square root of a negative number. As the square of either a positive or a negative number is itself positive, it follows that a negative number cannot have a *real* square root: the square root of a negative number or quantity is termed an *imaginary* quantity.

Thus $+4^2 = 16$ and $(-4)^2 = 16$

$$\therefore \quad \sqrt{16} = \pm 4$$

but $\sqrt{-16}$ is neither $+4$ nor -4. It is an imaginary number. As $\sqrt{-16}$ can be written as $\sqrt{16 \times (-1)}$ or $\sqrt{16} \times \sqrt{-1}$, we can write $\sqrt{-16} = 4\sqrt{-1}$.

If we write i for $\sqrt{-1}$, then $\sqrt{-16} = 4i$. This method of expression enables us to write down a solution, though imaginary, of a quadratic equation in which $4ac > b^2$.

Solve: $$x^2 + x + 1 = 0$$

Substituting $a = 1$, $b = 1$ and $c = 1$ in

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{-3}}{2}$$

As $\sqrt{-3}$ has no real value (since no number on being squared can give -3), let $i = \sqrt{-1}$ as before

$$\therefore \quad x = \frac{-1 \pm \sqrt{3 \times (-1)}}{2}$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2}$$

$$\therefore \quad x = \frac{-1 \pm 1{\cdot}732i}{2}$$

$$\therefore \quad x = -0{\cdot}5 \pm 0{\cdot}866i$$

4. A reference to the Graphical Solution of a Quadratic Equation (given in the First Year Course) will help to show the significance of these imaginary roots.

Fig. 1 shows the graph of $6x^2 + 11x - 10$; this function is equal to 0 when $x = \frac{2}{3}$ or $-2\frac{1}{2}$ *i.e.*, the roots of the equation $6x^2 + 11x - 10 = 0$ are $\frac{2}{3}$ and $-2\frac{1}{2}$ (both real).

Fig. 2 shows the graph of $5x^2 - 13x + 1$; the roots of the equation $5x^2 - 13x + 1 = 0$ are $0{\cdot}079$ and $2{\cdot}521$ (both real).

Fig. 3 is the graph of $x^2 + x + 1$, which does not cut the x axis: thus there is no real value of x satisfying the equation $x^2 + x + 1 = 0$. This position illustrates the case of a quadratic equation with imaginary roots.

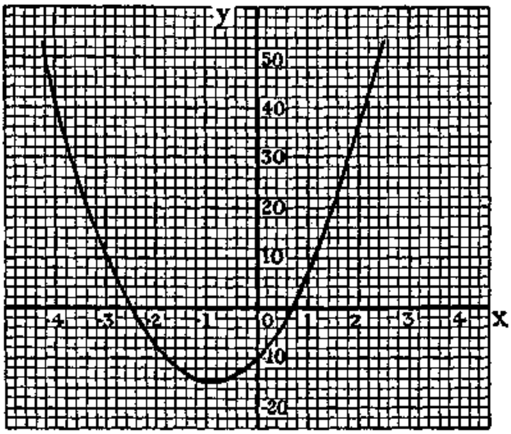

FIG. 1.

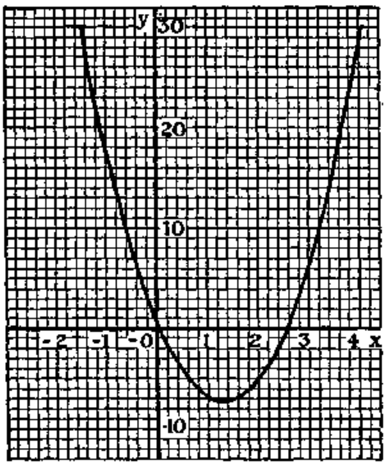

FIG. 2.

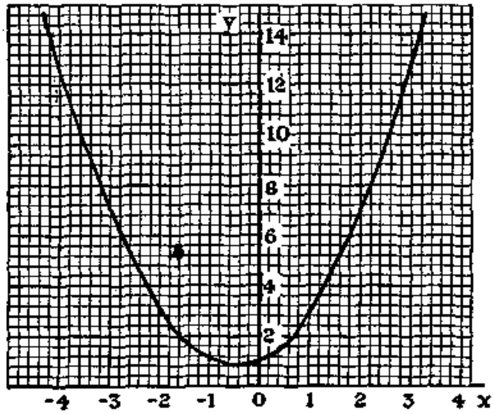

FIG. 3.

A reference to the graph of $x^2 - 6x + 9$ (Fig. 4) will serve as a reminder of the case in which a quadratic equation has two equal roots—when $b^2 = 4ac$. The equation

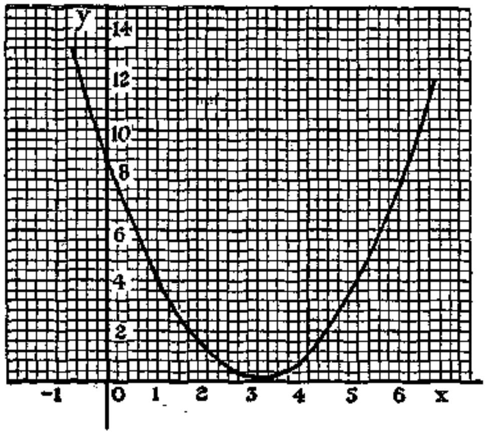

FIG. 4.

$x^2 - 6x + 9 = 0$ gives $(x - 3)(x - 3) = 0$, from which it is seen that $x = 3$ is the only root of the equation.

[The above curves are known as "parabolas."]

EXERCISE 4

Solve the equations (1–18).

1. $x^2 - 5x - 24 = 0$.
2. $6x^2 - 13x + 15 = 9$.
3. $3a^2 - 10 = a$.
4. $3y^2 - \frac{1}{8} = \frac{y}{4}$.
5. $3m^2 - m = 0$.
6. $10x - \frac{1}{x} = 3$.
7. $x^2 + 5x - 10 = 0$.
8. $3y^2 - 7y = 2$.
9. $0{\cdot}9(x + 1) = 0{\cdot}8 - x^2$.
10. $\dfrac{2}{3x + 1} + \dfrac{1}{2x + 1} = \frac{2}{7}$.
11. $\dfrac{x + 3}{4} - \dfrac{4}{x + 3} = 5$.
12. $0{\cdot}5 + \dfrac{2}{x + 3} - \dfrac{11}{x + 7} = 0$.
13. $5(x - 2)(x^2 - 2x - 3) = 0$. (N.C.T.E.C.)
14. $x^2 = x - 1$.
15. $5x^2 - 2x + 6 = 0$.
16. $3a^2 + 7a + 8 = 0$.
17. $m^2 + 10 = 6m$.
18. $3y^2 = 2y - 5$.
19. Find the factors of $x^2 + 1{\cdot}1x - 5{\cdot}22$.
20. Factorise $3x^2 - 2{\cdot}1x - 14{\cdot}82$.

21. The diagonal of a rectangle is 15 ins. long; the rectangle is 3 ins. longer than it is broad. Find its sides.

22. The area of an oblong is $89\frac{1}{4}$ sq. ins. and its length is 2 ins. more than its breadth. Find its sides.

23. When a chord of length l is drawn in a circle of radius r, it is known that $r = \dfrac{l^2 + 4h^2}{8h}$, when h is the maximum height of the arc cut off by the chord. Find h when $l =$ 12 ins. and $r =$ 10 ins.

24. If a stone is projected upwards with a velocity of v ft. per sec., its height h ft. after t secs. is given by

$h = vt - \frac{1}{2}gt^2$, where $g = 32$. After how many secs. will its height be 100 ft. if $v = 120$ ft. per sec.?

25. Given that $S = \frac{n}{2}\{2a + (n - 1)d\}$, find n when $S = 876$, $a = 2$ and $d = 3$.

26. The whole surface-area, S, of a solid cylinder whose height is h and radius r is given by $S = 2\pi r(h + r)$. Find r, when $S = 350$ sq. ins. and $h = 10$ ins.

27. Two aeroplanes pass over a town, one flying due E. at 80 m.p.h. and the other due S. at 60 m.p.h. The faster machine passes over at noon and the slower one $\frac{1}{4}$ hr. later. When will they be 150 mls. apart?

28. When a cable hangs between two supports whose distance apart is d, its length L, and the sag S are connected by the equation

$$L = \frac{8S^2}{3d} + d$$

Find d when $L = 90$ ft. and $S = 2\frac{1}{2}$ ft.

29. In the design of a reinforced-concrete beam a required dimension h is obtained from the formula:

$$\frac{2a}{bh} = \frac{E_c}{E_s} \times \frac{h}{d - h}$$

Solve this equation for h when $a = 1{\cdot}77$, $b = 8{\cdot}0$, $\frac{E_c}{E_s} = \frac{1}{15}$ and $d = 16$. (U.E.I., 1932.)

30. Another formula used in finding the strength of a concrete beam is $bn^2 + 2am(n - c) = 0$. Solve this for n when $b = 4{\cdot}5$, $a = 1{\cdot}7$, $c = 8$ and $m = 12{\cdot}5$.

31. The stretch x produced in a bar by a falling weight occurs in the following formula:

$$x^2 - 0{\cdot}001x - 0{\cdot}00203 = 0$$

Solve for x. (U.E.I., 1935.)

5. Simultaneous Quadratic Equations

Simultaneous Quadratic Equations are Simultaneous Equations (see Vol. I) which contain the unknown quantities in the second, but no higher degree.

The general treatment will be found in Vol. III. A few special cases only can be dealt with here.

(1) When one Equation is Linear

Using the linear equation, we can express one unknown quantity in terms of the other, and then substitute that value in the other (quadratic) equation. This will give a quadratic with one variable only, which can be solved in the usual way. Then the other variable can be found by substitution, as in the case of Simple Simultaneous Equations.

Examples

1. *Solve:*

$$\left.\begin{aligned} x + y &= 6 \\ x^2 + 3xy - y^2 &= 36 \end{aligned}\right\} \quad \begin{aligned} &\dots\dots (1) \\ &\dots\dots (2) \end{aligned}$$

From (1), $x = 6 - y$

$\therefore$ Substituting this value of x in (2), we get

$$(6 - y)^2 + 3(6 - y)y - y^2 = 36$$

$$\therefore \quad 36 - 12y + y^2 + 18y - 3y^2 - y^2 = 36$$

$$\therefore \quad -3y^2 + 6y = 0$$

$$\therefore \quad y^2 - 2y = 0$$

$$\therefore \quad (y)(y - 2) = 0 \qquad \therefore \quad y = 0 \text{ or } 2$$

Substituting $y = 0$ in (1), $x = 6$

,, $y = 2$,, (1), $x = 4$

$\therefore$ The required solutions are

When $x = 6 \quad y = 0$

,, $x = 4 \quad y = 2$

2. *Solve:*

$$\left.\begin{aligned} x^2 + 2y^2 &= 17 \\ x + 3y &= 9 \end{aligned}\right\} \quad \begin{aligned} &\dots\dots (1) \\ &\dots\dots (2) \end{aligned}$$

From (2) $\qquad x = 9 - 3y$

Substituting this value for x in (1), we get

$$(9-3y)^2+2y^2=17$$

$$\therefore\ 81-54y+9y^2+2y^2=17$$

$$\therefore\ 11y^2-54y+64=0$$

$$\therefore\ y=\frac{54\pm\sqrt{54^2-(4\times 11\times 64)}}{22}$$

$$\therefore\ y=\frac{54\pm 10}{22}=\tfrac{64}{22}\text{ or }\tfrac{44}{22}$$

$$\therefore\ y=\tfrac{32}{11}\text{ or }2$$

Substituting $y=\frac{32}{11}$ in (2)

$$x+\tfrac{96}{11}=9$$

$$\therefore\ x=\tfrac{3}{11}$$

Substituting $y=2$ in (2)

$$x+6=9$$

$$\therefore\ x=3$$

$\therefore$ The solutions are

$$x=\tfrac{3}{11},\ y=\tfrac{32}{11}$$

$$\text{and } x=3,\ y=2$$

(2) When both Equations are in the Second Degree

Solve: $xy=18,\ x^2-xy+y^2=67$

In this case, substitution as before would give an equation involving the fourth power of the variable, which would render the calculation more difficult and lengthy and usually not capable of solution. An easier method is as follows:

Since $x^2-2xy+y^2=(x-y)^2$

and $x^2+2xy+y^2=(x+y)^2$

it is easier to obtain the values of these expressions and so find the values of $x-y$ and $x+y$.

Thus,

$$\left.\begin{aligned} x^2 - xy + y^2 &= 67 \\ xy &= 18 \end{aligned}\right\} \text{subtract}$$

$$\overline{x^2 - 2xy + y^2 = 49}$$

$$\therefore \quad (x - y)^2 = 49$$

$$\therefore \quad x - y = \pm 7 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again,

$$\left.\begin{aligned} x^2 - xy + y^2 &= 67 \\ 3xy &= 54 \end{aligned}\right\} \text{add}$$

$$\overline{x^2 + 2xy + y^2 = 121}$$

$$\therefore \quad (x + y)^2 = 121$$

$$\therefore \quad x + y = \pm 11 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Using (1) and (2) in all possible ways

$$(a) \quad \left.\begin{aligned} x - y &= +\ 7 \\ x + y &= +\ 11 \end{aligned}\right\}$$

$$\therefore \quad 2x = 18$$

$$\therefore \quad x = 9 \quad \text{and } y = \frac{18}{x} = 2$$

$$(b) \quad \left.\begin{aligned} x - y &= -\ 7 \\ x + y &= +\ 11 \end{aligned}\right\}$$

$$2x = 4$$

$$\therefore \quad x = 2 \quad \text{and } y = \frac{18}{x} = 9$$

$$(c) \quad \left.\begin{aligned} x - y &= +\ 7 \\ x + y &= -\ 11 \end{aligned}\right\}$$

$$\therefore \quad 2x = -\ 4$$

$$\therefore \quad x = -\ 2 \quad \text{and } y = \frac{18}{x} = -9$$

$$(d) \quad \begin{aligned} x - y &= -\ 7 \\ x + y &= -\ 11 \end{aligned}$$

$$\therefore \quad 2x = -\ 18$$

$$\therefore \quad x = -\ 9 \quad \text{and } y = \frac{18}{x} = -2$$

Thus the solutions are

$$\left.\begin{array}{lrr} \text{When} & x = 9, & y = 2 \\ \text{,,} & x = 2, & y = 9 \\ \text{,,} & x = -2, & y = -9 \\ \text{,,} & x = -9, & y = -2 \end{array}\right\}$$

EXERCISE 5

Solve:

1. $\left.\begin{array}{l} x - y = 1 \\ x^2 + y^2 = 61 \end{array}\right\}$

2. $\left.\begin{array}{l} 3x - 2y = 7 \\ x^2 - 3xy + y^2 = -19 \end{array}\right\}$

3. $\left.\begin{array}{l} 4x + 5y = 0 \\ 2x^2 + xy - y^2 = 14 \end{array}\right\}$

4. $\left.\begin{array}{l} x + y + 1 = 0 \\ 3x^2 - 5y^2 - 7 = 0 \end{array}\right\}$

5. $\left.\begin{array}{l} x + y = 1 \\ 3x^2 - xy + y^2 = 37 \end{array}\right\}$

6. $\left.\begin{array}{l} 2x + 3y = 14 \\ 4x^2 + 2xy + 3y^2 = 60 \end{array}\right\}$

7. $\left.\begin{array}{l} x - 2y = 2 \\ xy = 12 \end{array}\right\}$

8. $\left.\begin{array}{l} xy = 4 \\ x^2 + y^2 = 17 \end{array}\right\}$

9. $\left.\begin{array}{l} 2xy = 80 \\ x^2 + y^2 = 89 \end{array}\right\}$

10. $\left.\begin{array}{l} xy + 3x = 15 \\ 2y + 3xy = 22 \end{array}\right\}$

(Hint—Multiply first equation by 3, subtract and substitute.)

11. $\left.\begin{array}{l} xy - 6y = 1 \\ 2xy + 4x = 10 \end{array}\right\}$

12. $\left.\begin{array}{l} x^2 + 2y = 1\frac{3}{4} \\ x^2 - x + y = \frac{1}{2} \end{array}\right\}$

13. Solve the following equations:

(i) $6x^2 + 8x - 9 = 0$

(ii) $\dfrac{2}{x-1} + \dfrac{1}{x-3} = \dfrac{2}{x}$.

(iii) $x^2 - 9y^2 = 24$
$x - 3y = 8$

(U.L.C.I., 1936.)

6. Graphical Solution of Simultaneous Quadratic Equations

In considering this method of solving Simultaneous Quadratics, it must be remembered that the co-ordinates of every point on a graph must satisfy the equation of that graph; therefore, if two graphs intersect, the co-ordinates of the point (or points) of intersection must satisfy both graphs simultaneously. If, therefore, we are seeking those x and y values which satisfy two equations at the same time, that is, we are seeking the roots of a pair of simultaneous equations, we must find the co-ordinates of the points of intersection of their graphs.

Consider a simple example.

Solve graphically $\left.\begin{aligned} 4x - y + 15 &= 0 \\ x^2 - y &= 0 \end{aligned}\right\}$ . . . . . (1) . . . . . (2)

Equation (1) can be written, $y = 4x + 15$; its graph is a straight line.

Equation (2) can be written $y = x^2$; its graph is a parabola.

The roots of the simultaneous equation given above will be the co-ordinates of the points where the straight line intersects the parabola.

Draw the graph of $y = x^2$ when x varies from -8 to $+8$.

Tabulating the corresponding values of x and y:

x . .	-8	-6	-4	-2	0	2	4	6	8
$y=x^2$.	64	36	16	4	0	4	16	36	64

The graph is shown in Fig. 5.

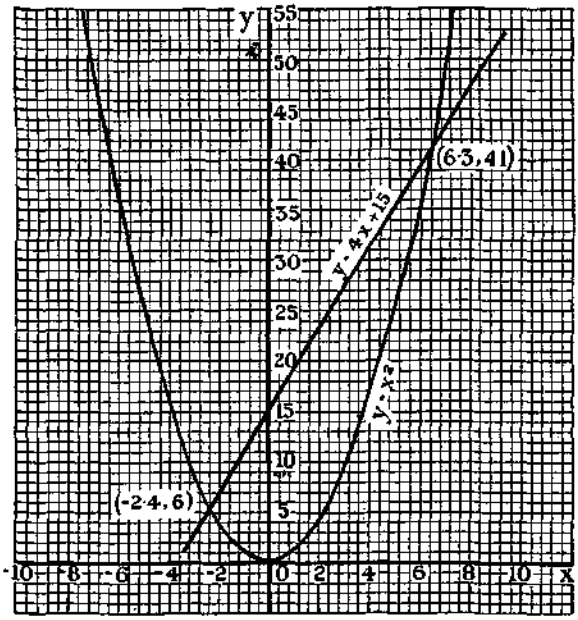

FIG. 5.

The line $y = 4x + 15$ can be drawn by taking two suitable points, such as:

x	-2	$+2$
$y = 4x + 15$.	$+7$	$+23$

As seen in the diagram, these graphs intersect at (approx.)

$$x = -2{\cdot}4, \qquad y = 6$$

and $$x = 6{\cdot}3, \qquad y = 41$$

$\therefore$ The roots of the simultaneous equation

$$\left.\begin{array}{l} 4x - y + 15 = 0 \\ x^2 - y = 0 \end{array}\right\} \text{are } (-2{\cdot}4,\ 6) \text{ and } (6{\cdot}3,\ 41)$$

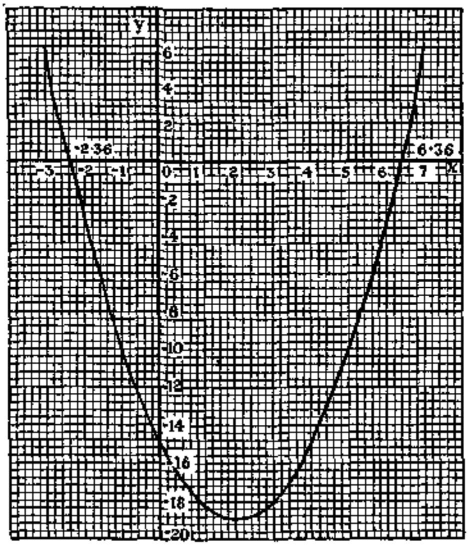

FIG. 6.

Checking algebraically, we get :

$$y = x^2 = 4x + 15$$

For the x roots of the equation, solve $x^2 = 4x + 15$

$$x^2 - 4x - 15 = 0$$

$$\therefore \quad x = \frac{4 \pm \sqrt{16 + 60}}{2} = 6{\cdot}36 \text{ or } -2{\cdot}36$$

and by substitution of these values in $y = 4x + 15$

$y = 4(6{\cdot}36) + 15 = 40{\cdot}44$ which agree with the

or $y = 4(-2{\cdot}36) + 15 = 5{\cdot}56$ approx. roots above.

It will also be noticed in the algebraic solution used above as a check that solving the simultaneous quadratic

$$4x - y + 15 = 0,\ x^2 - y = 0,$$

is the same problem as solving $x^2 = 4x + 15$ or $x^2 - 4x - 15 = 0$. We could have obtained the roots by drawing the graph of $x^2 - 4x - 15$ and noting the x values of the points where $x^2 - 4x - 15$ cuts the x axis—that is, the values of x when $x^2 - 4x - 15 = 0$. This graph is shown in Fig. 6.

7. Further Examples

1. *Solve graphically:*

$$\left.\begin{aligned} xy &= 20 \\ x^2 + y^2 &= 64 \end{aligned}\right\} \qquad \begin{aligned} &\ldots\ldots\ldots (1) \\ &\ldots\ldots\ldots (2) \end{aligned}$$

Before tabulating x and y values and drawing the graphs from them, consider equation (2), in which $x^2 + y^2 =$ a constant.

If AB (Fig. 7) represents a portion of a curve and P_1 is

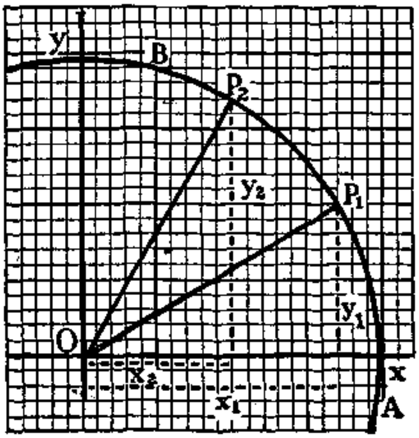

FIG. 7.

any point on the curve, then the co-ordinates of P_1 are x_1 and y_1 (say), where $x_1^2 + y_1^2 =$ a constant (in this case 64). But $x_1^2 + y_1^2 = (OP_1)^2$ $\therefore$ $(OP_1)^2 = 64$ and $OP_1 = \pm 8$.

Similarly if P_2 is another point on the same curve, its co-ordinates (x_2, y_2) must also satisfy the equation

$$x_2^2 + y_2^2 = 64$$

But $\quad x_2^2 + y_2^2 = (OP_2)^2 \qquad \therefore \quad (OP_2)^2 = 64$

$$\therefore \quad OP_2 = \pm 8.$$

Similarly for all other points on the curve: they will all lie on a curve whose distance from O is $\pm$ 8.

$\therefore$ The graph of $x^2 + y^2 = 64$ is a circle, whose centre is at O, and whose radius is 8 ins.

$\therefore$ To draw the graph of $x^2 + y^2 = 64$, we draw this circle of radius 8 ins. with centre at the origin, which is much easier and quicker than tabulating corresponding values of x and y.

To draw the graph of $xy = 20$,

$$y = \frac{20}{x}$$

Tabulating values from this, we get:

x . . .	-8	-6	-4	-2	0	2	4	6	8
$y = \frac{20}{x}$ . .	$-2\frac{1}{2}$	$-3\frac{1}{3}$	-5	-10	$-\infty$	10	5	$3\frac{1}{3}$	$2\frac{1}{2}$

On drawing this, which is called a Hyperbola, we find it cuts the circle at four points A, B, C, D (see Fig. 8).

At point A, $x = 2{\cdot}8$, $\quad y = 7{\cdot}4$.
,, ,, B, $x = 7{\cdot}4$, $\quad y = 2{\cdot}8$.
,, ,, C, $x = -7{\cdot}4$, $y = -2{\cdot}8$.
,, ,, D, $x = -2{\cdot}8$, $y = -7{\cdot}4$.

The actual roots found in the usual algebraic way agree very nearly with these.

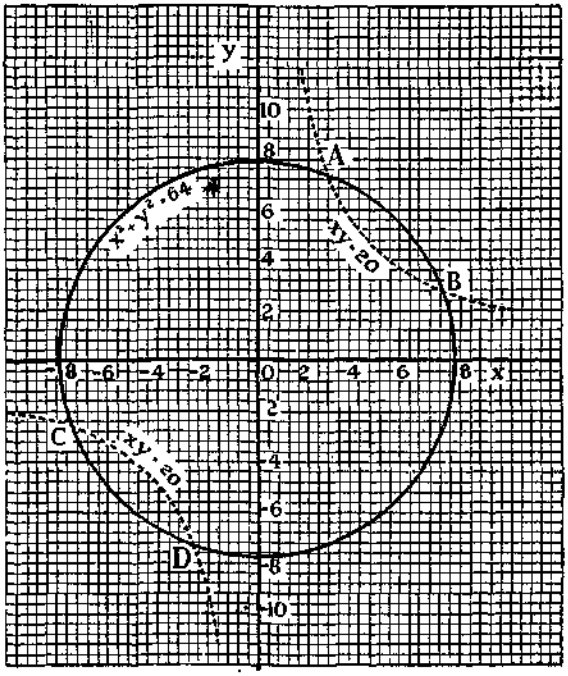

FIG. 8.

2. *Solve graphically :*

$$xy = 10 \qquad \dots \dots \dots \dots \quad (1)$$
$$y = x^2 + 6 \qquad \dots \dots \dots \dots \quad (2)$$

Plotting the graphs in the usual way, we find that (1) is a curve of the same type as (1) in the last example (a curve known as a hyperbola), while (2) is a parabola. In this case, there is only one point of intersection; its co-ordinates

are $x = 1{\cdot}35, y = 7{\cdot}75$. Thus the roots of the simultaneous quadratic are these values, as nearly as can be read directly from the graph (see Fig. 9).

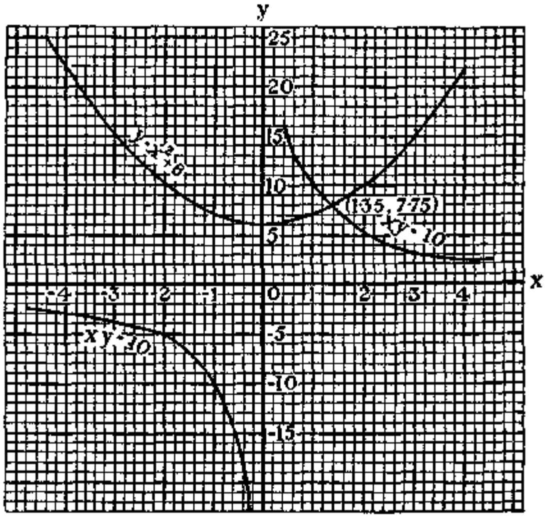

FIG. 9.

[If more accurate values are required, it would be necessary to plot the quadrant containing the point of intersection on a much larger scale, showing the portions of the curves lying in that quadrant.]

It will also be noticed that we have solved graphically an equation which would have proved difficult of algebraic solution: this is often the case: sometimes the graphical method is the *only* method of obtaining a solution.

The student should now solve a few of the equations given in Ex. 5, by the graphical method.

EXERCISE 6

1. (*a*) Solve the equation $x^2 + 0{\cdot}4x = 4{\cdot}37$.
   (*b*) Solve $x - y = 2{\cdot}70$, $xy = 0{\cdot}4275$.
   (U.L.C.I., 1927.)

2. (*a*) Given $10(7 - d) = 40d^2$, find the values of d.
   (*b*) Given the simultaneous equations,

$$0 = a + \frac{b}{400}, \quad 18 = a + \frac{b}{4a}$$

find the values of a and b. (U.L.C.I., 1928.)

3. Eliminate $\tan\theta$ from the equations (1) and (2) and show that $f^2 - pf - q^2 = 0$.

$$(1)\ f = p + \frac{q}{\tan\theta}$$

$$(2)\ q = \frac{f}{\tan\theta}$$

Obtain two values of f in terms of p and q, and give these values when $p = 4$ and $q = 2$. (U.L.C.I., 1935.)

4. Solve the equations

$$x^2 - 4y^2 = 8, \quad x - 2y = 4$$

(U.L.C.I., 1935.)

5. One of the values of x for which the function $2x^3 + x^2 - 13x + 6$ has a zero value is -3. Find the other two. (N.C.T.E.C., 1931.)

6. If $\dfrac{3c^2 + 3x_1c + x_1^2}{3c^2 + 3x_2c + x_2^2} = \dfrac{x_2V_1}{x_1V_2}$, find the positive value of C when $x_1 = 4$, $x_2 = 6$, $V_1 = 120$, $V_2 = 315$.
   (N.C.T.E.C., 1933.)

7. Find the values of the constants l, m, n in order that the function

$$[l(x - 3)(x - 4) + m(x - 4)(x - 2) + n(x - 2)(x - 3)]$$

may have the values 4, 1, 6, when x has the values 2, 3, 4

respectively. Having found the values of the constants, reduce the function to the form $ax^2 + bx + c$, and show by substitution that both forms give the same value when $x = 5$. (N.C.T.E.C., 1933.)

8. Using the same scales and axes, draw the curve whose equation is $x^2 + y^2 = 4$, and the line whose equation is $y = 2x + 1{\cdot}5$. State the values of x and y for the points where the curve and line intersect.

CHAPTER 4

MENSURATION

1. Plane Rectilinear Figures

The methods of finding the areas of squares, rectangles, parallelograms and triangles have previously been dealt with and the following rules established. (Vol. I.)

1. To find the area of any parallelogram (including the square and rectangle), multiply the length of one side by the length of the perpendicular to that side from the opposite parallel side. (Thus it will be noted that all parallelograms on equal bases and of equal heights perpendicular to the base, have the same area.)

2. To find the area of a triangle, find the area of the corresponding parallelogram by the above rule and divide by 2 (since the diagonal of a parallelogram divides it into two equal parts).

3. It was also shown that in the case of a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides (Principle of Pythagoras).

4. Another useful rule was established for finding the area of a triangle when two sides and the angle between them were given. The area is found by multiplying together the two sides and the sine of the contained angle.

Thus $$A = \tfrac{1}{2}ab \sin C$$

where A denotes the area, a and b the two sides and C the angle between them.

2. Area of Triangle when Three Sides are Given

Another useful rule can be obtained for finding the area of a triangle when the three sides are given, in which case the previous rules cannot be applied simply and directly.

Let the three sides of the triangle ABC be of lengths a, b and c respectively (Fig. 10).

From A drop AD perpendicular to BC.

Let AD $= h$ and BD $= x$.

Then DC $= a - x$.

Then, by the Principle of Pythagoras,

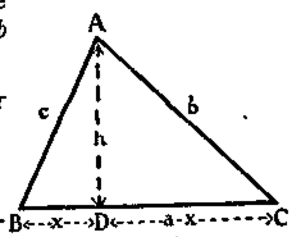

FIG. 10.

$$h^2 = c^2 - x^2 = b^2 - (a - x)^2$$
$$\therefore \quad c^2 - x^2 = b^2 - a^2 + 2ax - x^2$$
$$\therefore \quad c^2 = b^2 - a^2 + 2ax$$
$$\therefore \quad x = \frac{c^2 + a^2 - b^2}{2a}$$

But the area of the triangle $= \dfrac{\text{base} \times \text{height}}{2}$

$$\therefore \quad \text{Area} = \frac{a \times h}{2}$$

But $$h = \sqrt{c^2 - x^2}$$

$$\therefore \quad \text{Area} = \frac{a \times \sqrt{c^2 - x^2}}{2}$$
$$= \frac{a}{2} \times \sqrt{(c + x)(c - x)}$$

Now substitute $x = \dfrac{c^2 + a^2 - b^2}{2a}$ (as found above).

$$\therefore \text{ Area} = \frac{a}{2} \times \sqrt{\left\{c + \frac{c^2 + a^2 - b^2}{2a}\right\} \times \left\{c - \frac{c^2 + a^2 - b^2}{2a}\right\}}$$
$$= \frac{a}{2} \times \sqrt{\frac{2ac + c^2 + a^2 - b^2}{2a} \times \frac{2ac - (c^2 + a^2 - b^2)}{2a}}$$

$$= \frac{a}{2} \times \sqrt{\frac{(a^2 + 2ac + c^2) - b^2}{2a} \times \frac{b^2 - (a^2 - 2ac + c^2)}{2a}}$$

$$= \frac{a}{2} \times \sqrt{\frac{(a + c)^2 - b^2}{2a} \times \frac{b^2 - (a - c)^2}{2a}}$$

$$= \frac{a}{2}\sqrt{\frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4a^2}}$$

$$= \frac{a}{2 \cdot 2a} \cdot \sqrt{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}$$

$$= \tfrac{1}{4}\sqrt{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}$$

Now let $\quad s = \dfrac{a + b + c}{2}$

Then $a + b + c = 2s$

$$a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b)$$
$$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c)$$
$$b - a + c = a + b + c - 2a = 2s - 2a = 2(s - a)$$

Then by substituting these values in above :

$$\text{Area} = \tfrac{1}{4}\sqrt{(2s)(2s - 2b)(2s - 2c)(2s - 2a)}$$
$$= \tfrac{1}{4}\sqrt{(2s)(2)(s - b)(2)(s - c)2(s - a)}$$
$$= \tfrac{1}{4}\sqrt{(16(s)(s - a)(s - b)(s - c)}$$
$$= \tfrac{4}{4}\sqrt{(s)(s - a)(s - b)(s - c)}$$

$$\therefore \quad \textbf{Area} = \sqrt{(\boldsymbol{s})(\boldsymbol{s} - \boldsymbol{a})(\boldsymbol{s} - \boldsymbol{b})(\boldsymbol{s} - \boldsymbol{c})}$$

where a, b and c are the three sides and s = half the sum of the three sides.

Example

Find the area of a triangle whose sides are 7·3 *ins.*, 4·7 *ins. and* 5·4 *ins. respectively.*

Let A = area

Then $\quad A = \sqrt{(s)(s - a)(s - b)(s - c)}$

where $\quad a = 7{\cdot}3$ ins., $b = 4{\cdot}7$ ins., $c = 5{\cdot}4$ ins.

and

$$s = \frac{7{\cdot}3 \text{ ins.} + 4{\cdot}7 \text{ ins.} + 5{\cdot}4 \text{ ins.}}{2} = \frac{17{\cdot}4 \text{ ins.}}{2} = 8{\cdot}7 \text{ ins.}$$

$\therefore \quad A = \sqrt{(8{\cdot}7)(8{\cdot}7 - 7{\cdot}3)(8{\cdot}7 - 4{\cdot}7)(8{\cdot}7 - 5{\cdot}4)}$

$\therefore \quad A = \sqrt{8{\cdot}7 \times 1{\cdot}4 \times 4 \times 3{\cdot}3}$

Taking logs. of both sides,

$$\begin{aligned}\log A &= \tfrac{1}{2}\{\log 8{\cdot}7 + \log 1{\cdot}4 + \log 4 + \log 3{\cdot}3\}\\ &= \tfrac{1}{2}\{0{\cdot}9395 + 0{\cdot}1461 + 0{\cdot}6021 + 0{\cdot}5185\}\\ &= \tfrac{1}{2}\{2{\cdot}2062\}\\ &= 1{\cdot}1031\\ &= \log(12{\cdot}68)\end{aligned}$$

$\therefore \quad$ *$A = 12{\cdot}68$ sq. ins.*

3. Regular Polygons

Since a regular polygon can be divided into as many *equal* triangles as the figure has sides, the method of finding the area of the polygon consists of finding the area of one triangle and then multiplying by the suitable number.

Examples

1. *Find the area of a regular pentagon of side 6 ins.*

Let the pentagon ABCDE (Fig. 11) be divided into five equal triangles by joining each angular point to O, the centre of the circumscribed circle.

Draw ON perpendicular to CD.

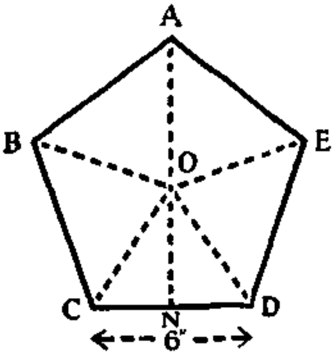

FIG. 11.

Then $\angle COD = \frac{1}{5} \times 360° = 72°$.

$\therefore \quad \angle OCD + \angle ODC = 180° - 72° = 108°$.

But $\angle OCD = \angle ODC$ (by symmetry)

$\therefore \quad \angle OCD = 54°$

$\therefore \quad \dfrac{ON}{CN} = \tan 54°$

$\therefore \quad ON = CN \,.\, \tan 54° = 3 \text{ ins.} \times \tan 54°$

But area of $\triangle OCD = \dfrac{CD \times ON}{2}$

$$= \frac{6 \times 3 \times \tan 54^\circ}{2} \text{ sq. ins.}$$

$$= (9 \tan 54^\circ) \text{ sq. ins.}$$

$\therefore$ Area of pentagon

$$= 5 \times 9 \times \tan 54^\circ \text{ sq. ins.}$$
$$= 45 \tan 54^\circ \text{ sq. ins.}$$
$$= \mathit{61{\cdot}93 \text{ sq. ins.}}$$

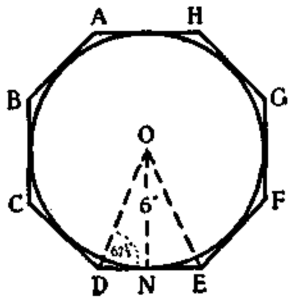

Fig. 12.

2. *A regular octagon circumscribes a circle of radius 6 ins. Find its area and perimeter.*

Let the octagon ABCDEFGH (Fig. 12) be divided into eight triangles each equal to $\triangle ODE$ (where O is the centre of the circle).

Then

$$\angle DOE = \frac{360^\circ}{8} = 45^\circ$$

$$\therefore \quad \angle ODE = \frac{180 - 45}{2} = \frac{135}{2} = 67\tfrac{1}{2}^\circ$$

From O, draw ON perpendicular to DE.

Then $\quad ON = 6$ ins.

But $\quad \dfrac{6}{DN} = \tan 67\tfrac{1}{2}^\circ \quad \therefore \quad DN = \dfrac{6}{\tan 67\frac{1}{2}^\circ}$

Area of $\triangle ODE = \dfrac{\text{base} \times \text{ht.}}{2} = \dfrac{2DN \,.\, ON}{2} = DN \,.\, ON$

$$= \frac{6 \times 6}{\tan 67\frac{1}{2}^\circ} = \frac{36}{\tan 67\frac{1}{2}^\circ}$$

$$\therefore \quad \text{Area of octagon} = 8 \times \frac{36}{\tan 67\frac{1}{2}^\circ}$$
$$= \frac{288}{\tan 67\frac{1}{2}^\circ}$$
$$= \mathit{119{\cdot}3\ sq.\ ins.}$$

$$\text{Perimeter of octagon} = 8 \times \text{DE}$$
$$= 16 \times \text{DN}$$
$$= \frac{16 \times 6}{\tan 67\frac{1}{2}^\circ}$$
$$= \frac{96}{\tan 67\frac{1}{2}^\circ}$$
$$= \mathit{39{\cdot}77\ ins.}$$

4. Irregular Rectilinear Figures

In the case of irregular polygons, the figure can be divided into triangles of unequal sizes, and the student must use his own ingenuity for finding the area of each particular triangle according to the data given. Methods which will be dealt with later, when finding areas of irregular figures with curved boundaries, may prove of use in some cases of figures with rectilinear edges.

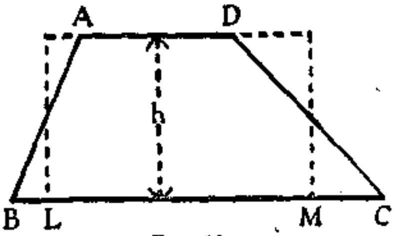

FIG. 13.

One particular case which is often useful is the method of finding the area of a trapezoid (sometimes called a trapezium)—a four-sided figure with two sides parallel (but unequal) (Fig. 13), and dealt with in Vol. I.

It will be sufficient to remind the student that the

$$\begin{aligned} \textit{Area of trapezoid } \mathrm{ABCD} &= \textit{average length} \times \textit{perp. height} \\ &= \tfrac{1}{2}(\mathrm{AD} + \mathrm{BC}) \times h \\ &= \mathrm{LM} \times h \end{aligned}$$

EXERCISE 7

1. Find the area of a triangle ABC, when AB = 14 ins., BC = 11 ins. and $\angle$ABC = 70°.

2. If the area of the triangle in No. 1 were 50 sq. ins., what would be the angle ABC?

3. If the area of a triangle is 100 sq. ins. and two of its sides are 21 ins. and 15 ins. respectively, find the angle between those two sides.

4. The sides of a triangle are 10 ins., 13 ins., 17 ins. respectively. Find its area.

5. Find the area of a triangle whose sides are 23·22, 31·18 and 40·04 chains respectively.

6. A triangle has sides of 8·21, 10·36 and 12·58 miles. Find its area.

7. The three sides of a triangle measure 22 ft. 6 ins., 20 ft. 5 ins. and 18 ft. 10 ins. respectively. Find its area in square feet.

8. Given that a triangle has three sides measuring $67\frac{1}{2}$ mls. 90 mls. and $112\frac{1}{2}$ mls. respectively, find its area.

9. A triangle, whose sides are 13·5 ins., 32·4 ins. and 35·1 ins. respectively, is made of material whose weight per square inch is 2·3 oz. Find its weight in lb.

10. A triangular sheet of metal whose sides are 31 ins., 74·4 ins. and 80·6 ins. respectively, weighs 82·2 lb. Find its weight in oz. per square inch.

11. Find the area of a regular pentagon whose side is 1 ft.

12. Find the area of a regular octagon of side 6 ins.

13. Find the perimeter and area of a regular hexagon inscribed in a circle of 10 in. radius.

14. A regular decagon is inscribed in a circle of radius 5 ins. Find its area and perimeter.

15. A regular octagon circumscribes a circle of radius 5 ins. Find its area and perimeter.

16. In a circle of 6 in. diameter is inscribed a regular pentagon. Find its area and its perimeter.

17. Find the area of the largest hexagonal shank that can be cut from a round bar of metal 2 ins. in diameter.

18. The section of an open water-channel is a trapezoid the vertical depth of which is 3 ft., and the breadth at the bottom 2 ft. The inclination of the sides to the horizontal is 60° (outward). Find the area and perimeter of the section. The hydraulic mean depth $H = \dfrac{\text{area of section of fluid}}{\text{wetted perimeter of channel}}$. Find H when the channel is running full of water.

(U.L.C.I., 1927.)

5. Areas of Irregular Curved Figures

When irregular figures have curved boundaries, various methods can be adopted for finding their areas.

1. *By Use of Squared Paper.*

A simple, though tedious, method is to copy the figure on squared paper and then to count the squares. Knowing the area of one square, it is a simple calculation to find the whole area.

2. *The Trapezoidal Rule.*

The method of using this rule consists of an application of the rule for finding the area of a trapezoid. A base line is drawn along the figure (preferably along its greatest length) and this line is then divided into a number of equal parts. At each division point, an ordinate is drawn to meet the curve (on both sides of the base-line, if necessary). The figure is by this means divided into a number of narrow strips whose ends are approximately straight lines (the more strips taken will make the ends approximate more and

more to straight lines). We can now apply the rule for the area of a trapezoid to each strip and so find the area of the figure.

Consider Fig. 14, which in practice may be only the upper part of a given figure. The method shown will suffice for both parts either together or separately.

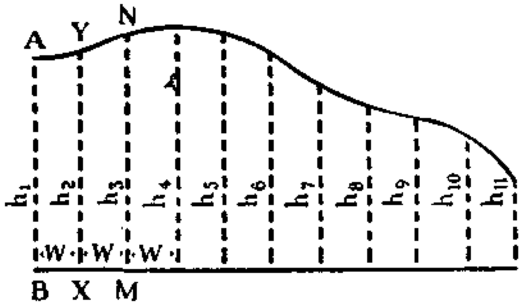

FIG. 14.

The base-line has been divided into ten equal parts and ordinates of lengths h_1, h_2, h_3 . . . h_{11} have been set up.

The area of the first strip BXYA is approximately that of a trapezoid whose parallel sides are h_1 and h_2 and whose ends are BX and AY (AY being considered as a straight line very approximately, especially if the width of the strip BX is small).

$\therefore$ Area of first strip BXYA $= \frac{h_1 + h_2}{2} \times w$ (where $w =$ BX).

Similarly area of second strip XMNY $= \frac{h_2 + h_3}{2} \times w$.

Similarly area of third strip $= \frac{h_3 + h_4}{2} \times w$, and so on.

$$\text{Area of last strip} = \frac{h_{10} + h_{11}}{2} \times w.$$

$$\therefore \text{ Total area} = w\left(\frac{h_1 + h_2}{2} + \frac{h_2 + h_3}{2} + \ldots + \frac{h_{10} + h_{11}}{2}\right)$$
$$= w\left(\frac{h_1 + 2h_2 + 2h_3 \ \ldots \ + 2h_{10} + h_{11}}{2}\right)$$
$$= w\left(\frac{h_1 + h_{11}}{2} + h_2 + h_3 + \ \ldots \ + h_{10}\right)$$

$\therefore$ *Area of figure shown is equal to the width of one division multiplied by the sum of half the first and last ordinates together with all the remaining ordinates.*

This is known as the Trapezoidal Rule.

Example

Find the area of the figure shown (Fig. 15) *by use of the Trapezoidal Rule.*

The ordinates are drawn at intervals of 10 mms. on a reduced scale and their heights measured as follows:

$h_1 = 14{\cdot}5$	mms.		$h_2 = 16{\cdot}5$	mms.
$h_{11} = 6$	,,		$h_3 = 20$	,,
———			$h_4 = 22$	,,
$20{\cdot}5$	,,		$h_5 = 21{\cdot}5$	,,
$\therefore \frac{1}{2}(h_1 + h_{11}) = 10{\cdot}25$	,,		$h_6 = 18$	,,
			$h_7 = 14$	,,
			$h_8 = 11$	,,
			$h_9 = 9$	,,
			$h_{10} = 8{\cdot}5$	,,
			$\frac{1}{2}(h_1 + h_{11}) = 10{\cdot}25$	,,

$\therefore$ Sum of $\frac{1}{2}$(first and last) and all other ordinates $\} = 150{\cdot}75$,,

$\therefore$ Area of figure $= 10 \times 150{\cdot}75$ sq. mms.
$= \mathit{1510}$ *sq. mms.* (approx.)

3. *The Mid-ordinate Rule.*

As this rule was dealt with in detail in the First Year Course, it will be sufficient to remind the student of the method.

As in the case of the Trapezoidal Rule, the base is divided into a suitable number of equal parts and ordinates are erected, in this case at the middle of each strip, such lines being known as mid-ordinates.

As each mid-ordinate represents the average height of the strip in which it stands, the area of the strip will be the product of the mid-ordinate and the width.

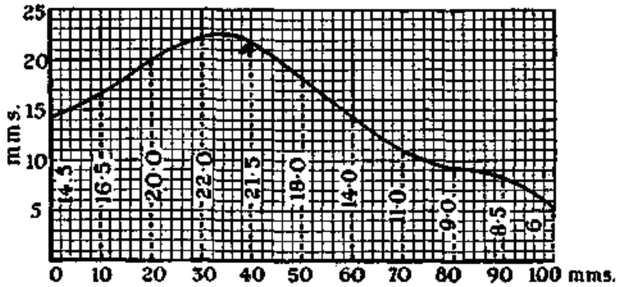

FIG. 15.

Thus the total area of the figure is the product of the average height of the mid-ordinates and the whole width of the figure.

4. *Simpson's Rule.*

This rule is the most accurate of the strip rules, and, though difficult to prove, not hard to memorise and to use.

In this case the base must be divided into an *even* number of equal parts, and ordinates erected at each point of division.

The rule then states :

$$Area = \frac{\textit{width of one strip}}{3} \times \textit{sum of} \left\{ \begin{array}{l} \textit{first and last ordinates,} \\ 4 \; (\textit{sum of even ordinates}), \\ 2 \; (\textit{sum of other odd ordinates}) \end{array} \right\}$$

Note

The sum of the *other* odd ordinates does not include the first and last as they have already been used.

Example

Use Simpson's Rule to find the area of the given figure (Fig. 16).

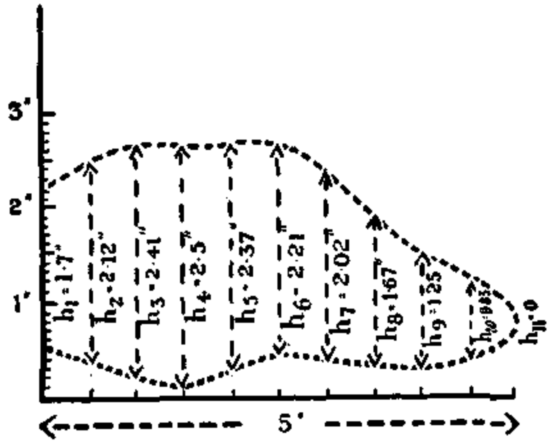

FIG. 16.

Having drawn and measured the ordinates (*i.e.*, the portions intercepted between the two curved lines, which will enable the calculation to be done in one operation, instead of two), we apply the rule :

Width of one division of base $= 0{\cdot}5$ in.

Sum of first and last ordinates $= 1{\cdot}7$ in. $+ 0 =$ *1·7 in.*

4 times the sum of even ordinates

$$= 4(2{\cdot}12 + 2{\cdot}5 + 2{\cdot}21 + 1{\cdot}67 + 0{\cdot}83)$$
$$= 4 \times 9{\cdot}33 = 37{\cdot}32 \text{ ins.}$$

2 times the sum of other odd ordinates

$$= 2(2{\cdot}41 + 2{\cdot}37 + 2{\cdot}02 + 1{\cdot}25)$$
$$= 2 \times 8{\cdot}05 = 16{\cdot}1 \text{ ins.}$$

$$\therefore \quad \text{Area} = \frac{0{\cdot}5 \text{ in.}}{3}\{1{\cdot}7 + 37{\cdot}32 + 16{\cdot}1\}$$
$$= \frac{0{\cdot}5 \times 55{\cdot}12}{3}$$
$$= 9{\cdot}2 \textit{ sq. ins.} \text{ (approx.)}$$

At this stage, the student would find it a useful exercise to cut out in cardboard or stiff paper a figure with a curved boundary and to copy it on paper; then to find its area by one of the previous methods. Copying the figure again on other sheets, he could find its area by the other methods and then compare the answers obtained.

6. Applications of Above Rules

Example 1

A point moves along a straight line so that its velocity, v, at any time, t, is as follows:

t (secs.)	*0*	*1*	*2*	*3*	*4*	*5*
v (ft. per sec.)	*7·6*	*10·7*	*13·8*	*16·9*	*20*	*23·1*

Find the average velocity from $t = 0$ to $t = 5$ secs. and the distance travelled in 5 secs.

The values of v and t when plotted give the straight line MN (Fig. 17).

$\therefore$ The average velocity will be the average height of the line MN.

This is evidently $\frac{1}{2}(7{\cdot}6 + 23{\cdot}1)$.

$$\therefore \quad \text{Average velocity} = \frac{30{\cdot}7}{2}$$
$$= 15{\cdot}35 \text{ ft. per sec.}$$

This is the velocity when $t = 2\frac{1}{2}$ secs. (*i.e.*, at the middle of the interval).

If the straight line PQ be drawn horizontally through $v = 15 \cdot 35$, it is evident that Q is as much below N as P is above M.

Also the point R is midway between P and Q.

$\therefore$ Area of $\triangle$PRM = area of $\triangle$QRN.

$\therefore$ Area between PQ and base = area between MN and base.

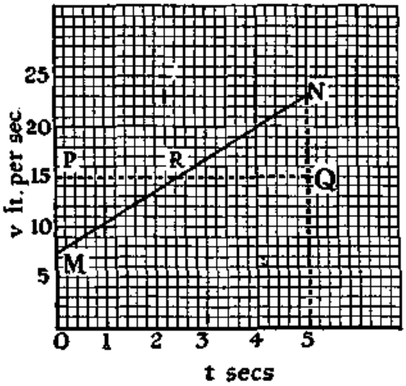

Fig. 17

But if the body travels 5 secs. with an average velocity of 15·35 ft. per sec., the distance covered is $15 \cdot 35 \times 5 = 76 \cdot 75$ ft., which is the area between PQ and the base.

But the area between MN and the base is the same (*i.e.*, 76·75 ft.).

$\therefore$ *The area between* MN *and the base represents the distance travelled in* 5 *secs.*

Example 2.

When the velocity-time graph is curved, the above method can also be applied as in the following example.

A body moves along a straight line so that its velocity, v, at any time, t, is given as follows:

t (secs.)	*0*	*1*	*2*	*3*	*4*	*5*
v (ft. per sec.)	*5*	*7·75*	*12*	*18*	*25*	*33·75*

Find the average velocity and the space travelled in 5 secs.

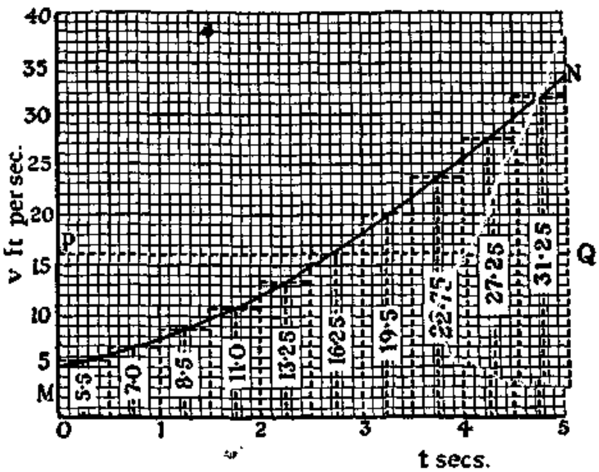

FIG. 18.

The graph is drawn as above (Fig. 18) and divided into a suitable number of strips (10) of equal width. Then each strip is treated as if its top were a straight line (*i.e.*, as if during the short interval included between the ordinates the velocity increased uniformly as in Ex. 1). This would be even more nearly true if 20 strips had been taken instead of 10. Now treating each strip as in Ex. 1 we can set up mid-ordinates in each strip and draw a horizontal line through the top of each one.

Then the average velocity is the average height of these mid-ordinates.

Also, the distance travelled during the interval of time represented by the width of a strip is the area between the portion of the curve at the top and the base, or, what is equivalent to it, the area between the horizontal line through the top of the mid-ordinate and the base.

Thus the total distance travelled in 5 secs. is the total area under the curve, and this is the product of the average value of the mid-ordinates (*i.e.*, of the velocity) and the time measured along the base. If the mid-ordinates are measured on the graph their heights are found to be :

5·5, 7·0, 8·5, 11·0, 13·25, 16·25, 19·5, 22·75, 27·25, 31·25

$$\therefore \text{ Average height of mid-ordinate} = \frac{162 \cdot 25}{10} = 16 \cdot 2 \text{ (app.)}$$

$\therefore$ *Average velocity during 5 secs. = 16·2 ft. per. sec.* (app.).

This, of course, would imply that a horizontal line drawn at a height of 16·225 represents the average velocity—see dotted line PQ. It will be noted that this is not the average of the initial and final velocities : neither does it coincide with the actual velocity at the middle of the interval of 5 secs. ; but it does imply that the areas between PQ and MN are equal although their shapes are slightly different.

Also the distance travelled in 5 *secs.* $= 16 \cdot 225 \times 5 \doteq 81 \cdot 1$ *ft.* (app.).

It would be interesting to find the area under the curve by another of the previous rules. The area would give the distance travelled, and this answer divided by 5 would give the average velocity during the 5 secs. interval.

(Simpson's Rule gives an area of 81·54.)

7. Work Done by a Force

Another interesting application is found in the case of the work done by a force.

The work done by a uniform force of P lb. when its point of application moves through a distance S ft. in its own direction is PS ft.-lb., which can be represented by a rectangle of length S ft. and height P lb. (in suitable units).

If, however, the force is variable, the resulting PS graph yields an irregular figure, and the previous method will enable us to see that the area under the curve will represent the work done and also to find both the value of the average force and the work done.

Example 3

If P *is the pressure of steam on the piston of a steam-engine at distance* S *from the beginning of the stroke, find the average pressure and the work done in a forward movement of* 16 *ins.*

S (*ins.*)	0	2	4	6·5	8	10·5	13	15	16
P (*thousand lb.*)	25	27·1	26·5	21	19·2	13	10	7·2	0

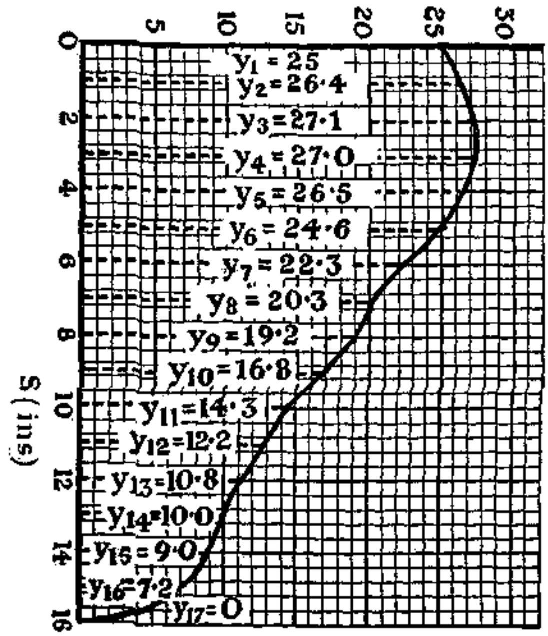

FIG. 19.

The figure shows the graph obtained (Fig. 19).

Area under curve, etc., by Mid-ordinate Rule.

The mid-ordinates drawn at 1, 3, 5, etc., are 26·4, 27, 24·6, 20·3, 16·8, 12·2, 10, 7·2.

$\therefore$ Average height of the mid-ordinates $= \frac{144 \cdot 5}{8} = 18 \cdot 06$

$\therefore$ Average pressure $= 18 \cdot 06$ thousand lb.
$= 18100$ lb. (approx.)

$\therefore$ Work done during 16 ins. movement

$$\doteq \frac{18100 \times 16}{12} \text{ ft.-lb.}$$

$= $ *24100 ft.-lb.* (approx.)

Alternative Method Using Simpson's Rule for Area

Setting up ordinates at 0, 1, 2, 3, 4, 5, 6, etc., and calling them y_1, y_2, y_3, etc., respectively, the following result is obtained (a larger scale graph was used).

$y_1 = 25$	$y_2 = 26 \cdot 4$	$y_3 = 27 \cdot 1$
$y_{17} = 0$	$y_4 = 27 \cdot 0$	$y_5 = 26 \cdot 5$
	$y_6 = 24 \cdot 6$	$y_7 = 22 \cdot 3$
25	$y_8 = 20 \cdot 3$	$y_9 = 19 \cdot 2$
	$y_{10} = 16 \cdot 8$	$y_{11} = 14 \cdot 3$
	$y_{12} = 12 \cdot 2$	$y_{13} = 10 \cdot 8$
	$y_{14} = 10$	$y_{15} = 9$
	$y_{16} = 7 \cdot 2$	129·2
	144·5	2
	4	258·4
	578·0	

$$\therefore \text{ Area} = \frac{\text{width of 1 strip}}{3}\left\{\begin{array}{l}\text{sum of first and last} + \\ 4\,(\text{sum of even ordinates}) + \\ 2\,(\text{sum of other odd ordinates})\end{array}\right\}$$

$$= \tfrac{1}{3}\{25 + 578 + 258 \cdot 4\}$$

$$= \frac{861 \cdot 4}{3} = 287 \text{ (approx.)}$$

$\therefore$ Work done = *287000 in.-lb.* (approx.)

$\therefore$ Work done $= \dfrac{287000}{12}$ ft.-lb.

$= $ *24000 ft.-lb.* (approx.)

Also the average pressure during 16 ins. movement

$$= \frac{287000}{16} \text{ in.-lb.}$$

$= $ *18000 lb.* (approx.)

These results agree with those previously found as closely as can be expected when different methods are used.

8. Volumes of Irregular Solids

At this stage, it will be convenient to note that the *volumes* of irregular (or non-symmetrical) solids can be found by the above methods.

The volume of a prism is the product of the end-area and its length, which can be represented by an oblong whose sides are the end-area and the length respectively.

If the cross-sectional area of the solid is not uniform, we can imagine the solid is cut up into many narrow slabs all parallel to one another.

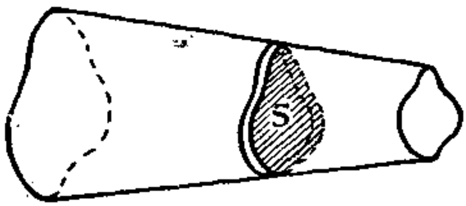

FIG. 20.

Consider the solid shown (Fig. 20). Its volume is the same as the total volume of all such slabs as S added together. But the volume of S is the product of its face-area and its thickness.

If the thickness of the slab is very small, we can suppose that both faces of the slab have the same area; if not, we

must take the average of those areas and then multiply by the thickness in order to find the volume.

The total of all such products will give the volume of the whole solid.

If we are given the areas of sufficient slabs in succession from one end of the solid to the other, we can plot these areas against the distances from one end. Since each

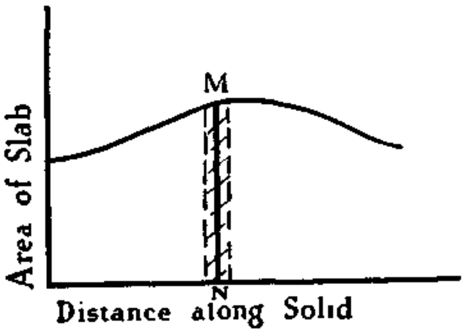

FIG. 21.

ordinate such as MN (Fig. 21) represents an area, we can look upon the rectangle (shaded) of which MN is the middle height as representing the area multiplied by a small thickness of the solid : *i.e.*, the volume of one slab.

Thus the total area under the curve will represent the sum of all such volumes, or the total volume of the solid.

Example

A piece of timber has a cross-sectional area, A, *perpendicular to its length at distance, d, from its end. Find the volume and average cross-sectional area.*

d (ft.) . . .	0	2	4	6	8	10	12
A (sq. ft.) . .	2·8	4·7	6·2	7·3	7·1	6·7	6·2

The graph is drawn with A as ordinates and d as abscissæ, and treated as in the previous examples.

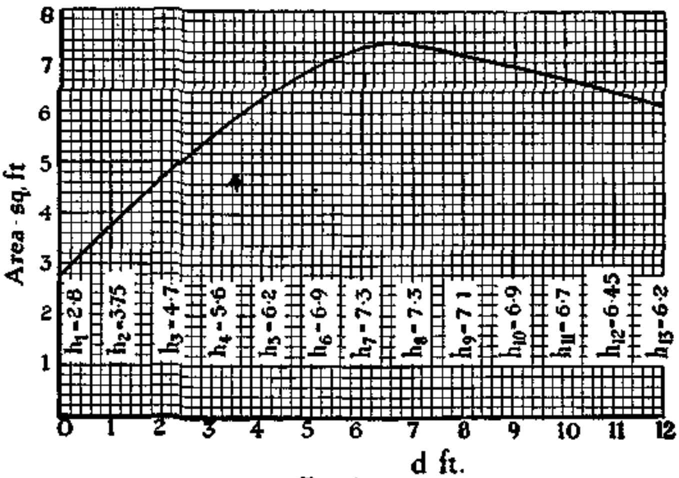

FIG. 22.

Set up and measure the ordinates h_1, h_2, h_3 . . . h_{13} and then apply Simpson's Rule (Fig. 22).

$h_1 = 2{\cdot}8$	$h_2 = 3{\cdot}75$	$h_3 = 4{\cdot}7$
$h_{13} = 6{\cdot}2$	$h_4 = 5{\cdot}6$	$h_5 = 6{\cdot}2$
$9{\cdot}0$	$h_6 = 6{\cdot}9$	$h_7 = 7{\cdot}3$
	$h_8 = 7{\cdot}3$	$h_9 = 7{\cdot}1$
	$h_{10} = 6{\cdot}9$	$h_{11} = 6{\cdot}7$
	$h_{12} = 6{\cdot}45$	$32{\cdot}0$
	$36{\cdot}90$	2
	4	$64{\cdot}0$
	$147{\cdot}60$	

$$\therefore \quad \text{Area} = \tfrac{1}{3} \times \{9 + 147{\cdot}6 + 64\}$$
$$= \frac{220{\cdot}6}{3} = \mathit{73{\cdot}5}$$

Volume of the timber = *73·5 cub. ft.*

$$\text{Average cross-section} = \frac{73{\cdot}5}{12} = \mathit{6{\cdot}13\ sq.\ ft.}$$

EXERCISE 8

1. The lengths of nine equidistant ordinates of a curve are 8, 10·5, 12·3, 11·6, 12·9, 13·8, 10·2, 8 and 6 ins. respectively; and the length of the base is 24 ins. Find the area between the curve and the base.

2. The ordinates of a curve are 2·3, 3·8, 4·4, 6·0, 7·1, 8·3, 8·2, 7·9, 6·2, 5·0 and 3·9 ins., and the interval between them is 1 in. Find the area under the curve, and also the average height of the curve.

3. Find the average value of y with respect to x, given the following values:

x . . .	0	0·5	1	1·5	2	2·5	3	3·5	4	4·5	5·0
y . . .	3	3·16	3·55	4·08	4·64	3·56	5·25	5·2	4·82	4·0	3·0

4. Find the average width of a circle, measured parallel to a fixed diameter, 10 cm. long.

5. The pressure (p) on the piston in a low-pressure cylinder of an engine is connected with the distance (s) from one end, during a stroke of 12 ins. as shown below. Find the average pressure and the work done.

s (ins.)	0	0·5	1·2	2·5	4·6	6·5
p (lb. per. sq. in.)	8	14	20·4	22	15·8	9

s	8·1	10	11·3	12
p	5·9	4·5	3·2	0

6. The speed of a car is v ft. per sec., t secs. after starting. Find the average speed and the distance travelled in the minute, given the following:

t (secs.)	0	10	20	30	40	50	60
v (ft. per sec.)	0	21	31·5	38	43	46·5	48·5

7. When a reservoir is filled to a height h ft. measured above the lowest part of its bed, the area, A, of the water-surface is as follows:

h (ft.)	12	25	40	65	80
A (sq. ft.)	36,000	67,000	74,000	80,000	84,000

Find the number of gallons in the reservoir when it is filled to a depth of 80 ft., if 1 gallon = 0·16 cub. ft.

8. A sq. yds. is the area of the section of a railway embankment at a distance x yds. from one end. Find the volume of earth in it.

x (yds.)	0	10	20	40	50	80	100
A (sq. yds.)	0	7	14	25	31	28	17

). A piece of oak has a cross-sectional area A sq. ft. at tance x ft. from one end. Find its weight if 1 cub. ft. ghs 50 lb.

(ft.) . .	0	2	5	7	10
(sq. ft.) . .	1·2	1·9	2·0	1·8	1·5

10. When a ship is drawing x ft. of water, the area of the vater-plane (the cross-section of the ship at the water-urface) is A sq. ft.

x (ft.) . .	3	5	7	9	12
A (sq. ft.) . .	3150	3900	4200	4350	4500

The displacement of water in tons is equal to the weight of the vessel in tons. If 1 ton of sea-water measures 35 cub. ft., what weight is loaded into the vessel when the draught of water increases from 3 ft. to 12 ft.?

11. The area of the cross-section of a ship's coal-bunker is A sq. ft. at a distance x ft. from one end. Values of A and x are:

x	0	15	30	45	60
A	85	120	131	130	115

Plot A vertically and x horizontally. Estimate in cubic feet the volume of the bunker the total length of which is 60 ft. How many tons of coal will the bunker hold if 1 ton occupies 45 cub. ft.? (U.E.I., 1932.)

12. A horizontal line to represent 5 ft. 9 ins. is set off and divided into eight equal parts. Starting at one end, perpendiculars are erected of lengths 38, $30\frac{1}{2}$, 26, 22, 19, 16, $12\frac{1}{2}$, 8 and 0 ins. This figure is the shape of a piece of sheet

copper used to cover a turret roof. Sketch the figure and, using the mid-ordinate rule, calculate

(i) the area in square feet,
(ii) the weight of copper to nearest oz. required, allowing 8% for laps, at 1 lb. 4 oz. per square foot. (U.E.I., 1935.)

13. L, M, N are points on an axis OX, on the same side of O, such that OL = 1 in., OM = 4 ins., ON = 7 ins. LP, MQ, NR are ordinates perpendicular to the axis OX, such that LP = 8 ins., MQ = 7 ins., NR = 4 ins. Prove $OP^2 = OQ^2 = OR^2$. Find the lengths of the ordinates at the mid-points of LM and MN of the circular arc through P, Q, R, and by means of Simpson's Rule and the five ordinates, estimate the area bounded by LP, NR, LN and the arc PQR. (N.C.T.E.C., 1935.)

CHAPTER 5

MENSURATION (*continued*)

THE CIRCLE

1. Previous Knowledge Revised

The following relations have already been dealt with in the First-Year Course :

(*a*) *Circumference* $= \pi \times$ *diameter*. $C = \pi d = 2\pi r$

(*b*) *Area* $= \pi r^2 = \dfrac{\pi d^2}{4} = 0{\cdot}7854d^2$

(*c*) *Area of Annulus* $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

$$= \pi(R + r)(R - r) \quad . \quad . \quad . \quad . \quad (1)$$

$$= \frac{\pi D^2}{4} - \frac{\pi d^2}{4}$$

$$= \frac{\pi}{4}(D + d)(D - d) \quad . \quad . \quad . \quad . \quad (2)$$

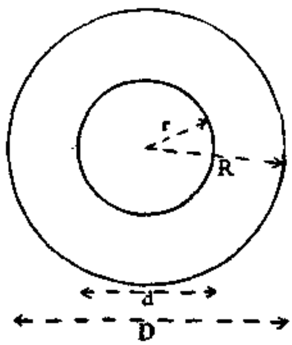

FIG. 23.

[where R, D refer to the radius and diameter of outer ring and r, d refer to the radius and diameter of inner ring (Fig 23)].

The usefulness of the forms (1) and (2) is shown when logarithms are used in calculations dealing with the annulus.

Example

Find the weight of 200 washers of thickness 3 mms. if the outside diameter of each is 4·5 cms. and the diameter of the hole in the centre is 1·75 cms. The density of the metal is 7·2 gms. per c.c.

$$\text{Area of one washer} = \frac{\pi}{4}(4{\cdot}5)^2 - \frac{\pi}{4}(1{\cdot}75)^2 \text{ sq. cms.}$$

$$= \frac{\pi}{4}(4{\cdot}5^2 - 1{\cdot}75^2) \text{ sq. cms.}$$

$$= \frac{\pi}{4}(4{\cdot}5 + 1{\cdot}75)(4{\cdot}5 - 1{\cdot}75) \text{ sq. cms.}$$

$$= \frac{\pi}{4} \times 6{\cdot}25 \times 2{\cdot}75 \text{ sq. cms.}$$

$$\therefore \text{ Volume of one washer} = \frac{\pi}{4} \times 6{\cdot}25 \times 2{\cdot}75 \times 0{\cdot}3 \text{ c.c.s.}$$

$\therefore$ Volume of 200 washers

$$= \frac{\pi}{4} \times 6{\cdot}25 \times 2{\cdot}75 \times 0{\cdot}3 \times 200 \text{ c.c.s.}$$

$\therefore$ Weight of 200 washers

$$= \frac{\pi}{4} \times 6{\cdot}25 \times 2{\cdot}75 \times 0{\cdot}3 \times 200 \times 7{\cdot}2 \text{ gms.}$$

$$= 5830 \text{ gms.}$$

$$= \mathit{5{\cdot}83\ kgs.}$$

It should be noted how the factorisation of $4{\cdot}5^2 - 1{\cdot}75^2$ reduced the calculation to multiplication. The student is also advised to leave unworked as much multiplication and division as possible, till all the operations are included and then to make one comprehensive calculation by means of logarithms. When this is possible, it is much neater and safer than making a series of minor calculations for the different steps in the problem.

2. Chords and Arcs of Circles

Before discussing the various numerical relations between these, one important geometrical proposition relating to chords in a circle must be mentioned. A formal proof can be found in any book of Euclidian Geometry.

The proposition states that if two chords intersect, either within or without the circle, the rectangles formed by their segments are equal in area.

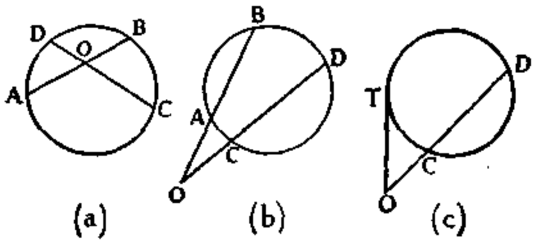

FIG. 24.

Thus in Fig. 24 (*a*) and (*b*),

$$AO \times OB = CO \times OD$$

Also if OAB in (*b*) be rotated about O, until A and B coincide at T,

then $$AO \times OB = (OT)^2$$
$$\therefore \quad (OT)^2 = CO \times OD,$$

where OT represents the tangent from O.

3. Length of Chord and Maximum Height of Arc.

PRQ is an arc of a circle (Fig. 25), whose centre is O, and of radius r. PSQ is the chord of length $2a$. RS is the maximum height (h) of the arc.

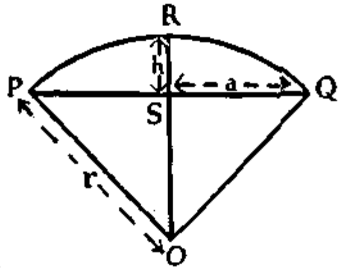

FIG. 25.

(*a*) *To find an expression for the length* PSQ *in terms of h and r.*

In $\triangle$OQS,

$$r^2 = (\text{OS})^2 + a^2 \text{ (by Principle of Pythagoras)}$$
$$= (r - h)^2 + a^2$$
$$= r^2 - 2hr + h^2 + a^2$$
$$\therefore \quad a^2 = 2hr - h^2$$
$$\therefore \quad \textit{Length of } \text{PSQ} = 2\sqrt{2hr - h^2}$$

(*b*) *To express r in terms of a and h.*

From above $\quad a^2 = 2hr - h^2$

$$\therefore \quad 2hr = a^2 + h^2$$
$$\therefore \quad r = \frac{a^2 + h^2}{2h}$$

(*c*) *To express h in terms of r and a.*

From above $\quad a^2 = 2hr - h^2$

$$\therefore \quad h^2 - 2hr + a^2 = 0$$

which, being solved for h, gives:

$$h = \frac{2r \pm \sqrt{(2r)^2 - 4a^2}}{2}$$
$$= \frac{2r \pm \sqrt{4r^2 - 4a^2}}{2}$$
$$= \frac{2r \pm 2\sqrt{r^2 - a^2}}{2}$$
$$\therefore \quad h = r \pm \sqrt{r^2 - a^2}$$

This shows that for the chord $2a$, in a circle of radius r, there are two values of h: that is, there are two possible

positions of the chord of length $2a$—one in which the height of the arc is

$$r + \sqrt{r^2 - a^2}$$

(see lower position in Fig. 26) and another in which the height is

$$r - \sqrt{r^2 - a^2}$$

(see upper position in Fig. 26).

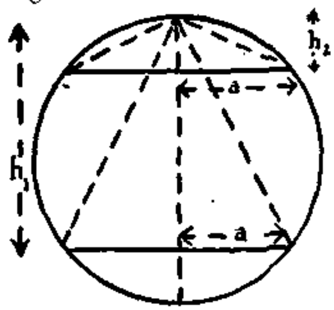

FIG. 26.

(*d*) *To express h in terms of r and a* (*Alternative Method*).

Since AO . OB = CO . OD (Fig. 27) and AO = OB.

$$\therefore \quad OB^2 = CO \cdot OD$$

$$\therefore \quad a^2 = h \times (2r - h)$$

$$\therefore \quad a^2 = 2rh - h^2$$

$\therefore$ As before, $\qquad h = r \pm \sqrt{r^2 - a^2}$.

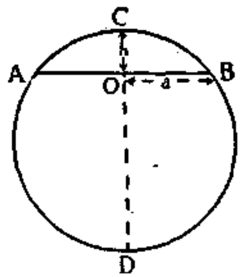

FIG. 27.

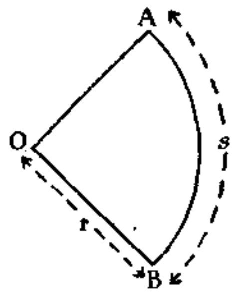

FIG. 28.

4. Length of an Arc

Let s be the length of the arc AB of a circle of radius r with centre O (Fig. 28).

Then s is the same fraction of the whole circumference as $\angle$AOB is of 360°.

$$\therefore \quad \frac{s}{\text{circum.}} = \frac{\text{angle subtended at centre}}{360^\circ}$$

$$\therefore \quad s = 2\pi r \times \frac{\text{angle at centre}}{360^\circ}$$

(the angle at the centre being expressed in degrees).

The length of s can also be expressed in terms of the radius r and the angle AOB expressed in radians. For, since by definition

1 radian = length of arc cut off by the radius

$$\therefore \quad \text{Number of radians in } \angle\text{AOB} = \frac{\text{AB}}{r} = \frac{s}{r}$$

$\therefore$ If θ be the number of radians in $\angle$AOB,

$$\frac{s}{r} = \theta$$

$$\therefore \quad s = r\theta \textit{ where } \theta \textit{ is in radians.}$$

5. Sector of a Circle

A sector is a portion of a circle bounded by two radii and the arc joining their extremities.

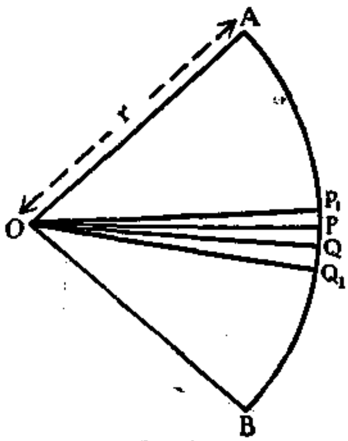

FIG. 29.

The area of a sector can be found by using the rule for the area of a triangle.

If the arc be divided into many small pieces such as PQ, and radii drawn as shown (Fig. 29), the sector will be divided into a large number of very narrow triangles; the more divisions there are in AB, the more nearly will PQ, QQ_1, etc., approximate to straight lines, and the sectors OPQ, etc., to triangles, whose heights are r.

$$\text{Area OPQ} = \frac{\text{PQ} \times r}{2}$$

$$\text{Area OPP}_1 = \frac{\text{PP}_1 \times r}{2}, \text{ etc., etc.}$$

$$\therefore \quad \text{Total area of sector} = \frac{(\text{PQ} + \text{PP}_1 + \text{QQ}_1 + \ldots) \times r}{2}$$

$$= \frac{\text{length of arc AB} \times r}{2}$$

$$= \frac{s \times r}{2}$$

But $\quad s = r\theta$ (where θ is angle at centre in radians)

$$\therefore \quad \text{Area of sector} = \frac{r\theta \times r}{2}$$

$$\therefore \quad \textit{Area of sector} = \tfrac{1}{2}r^2\theta$$

Note.—If $\theta = 2\pi$, the area of the whole circle $= \frac{1}{2}r^2 . 2\pi = \pi r^2$.

6. Area of Segment of Circle

A segment of a circle is the part of the circle lying between a chord and the corresponding arc.

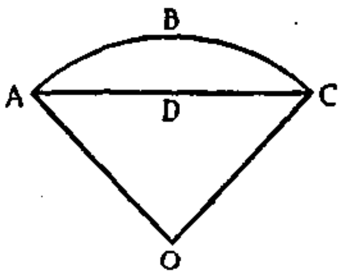

FIG. 30.

In Fig. 30, ABCD is a segment of circle whose centre is O. From the figure it is obvious that

Area of segment ABCD = *area of sector* ABCO − *area of* Δ ACO

7. Angular Velocity

The subject of Angular Velocity is so closely connected with the previous work on arcs, that it is desirable to consider the subject at this stage.

As previously shown, a radian is the angle subtended at the centre of a circle by an arc of the circumference equal to the radius.

In Fig. 31, if r be measured *along* the circumference as shown, $\angle AOB = 1$ radian, $\angle AOC = 2$ radians, $\angle AOD = 3$ radians.

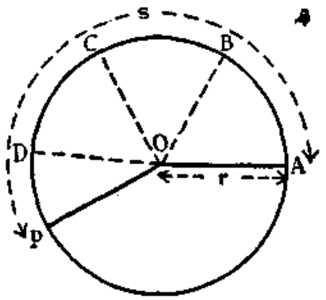

Fig. 31.

Also for any length s, measured *on* the circumference, the relation $s = r\theta$ is true, as previously shown.

Now imagine a body initially at A moving at the end of a rotating radius along the circumference past B and C.

The actual distance it would travel in 1 sec. along the circumference is its velocity or speed (speed is the better term). As this speed denotes movement along a line, it is often called " linear speed," and is generally denoted by v.

As A is moving along the circumference, the radius r is sweeping round and the number of radians through which it rotates in 1 sec., is called its " angular velocity " (ω). Evidently both A and the radius have the same angular velocity.

Now, suppose that the radius moves round so as to bring A to P in 1 sec.

Then s represents the linear speed (v) of A in ft. per sec. and $\angle AOP$, in radians, represents its angular velocity ω.

But, as shown previously, $s = r\theta$.

$$\therefore \quad v = r\omega$$

$\therefore$ The linear speed of a body rotating round a circle is equal to the radius multiplied by the angular velocity.

$\left(\text{We can also write} \qquad \omega = \frac{v}{r}\right)$

Example 1

A circular arc has a base of 4 ins. and a maximum height of 1·6 ins. Find (a) radius, (b) length of arc, (c) height of arc at point 1 in. from end of base, (d) area of segment (Fig. 32).

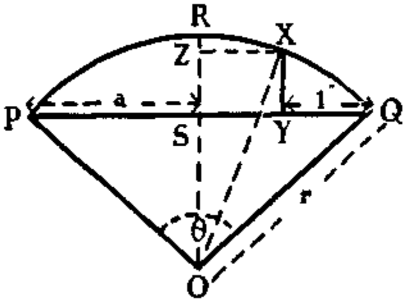

FIG. 32.

Given $\qquad$ SP = SQ = 2 ins.
RS = 1·6 ins.

(a) *Find radius r.*

Applying formula $\quad r = \dfrac{a^2 + h^2}{2h}$ . . . . (see p. 86).

$$r = \frac{2^2 + 1{\cdot}6^2}{2 \times 1{\cdot}6} = \frac{4 + 2{\cdot}56}{3{\cdot}2} = 2{\cdot}05 \textit{ ins.}$$

(b) *Find length of arc.*

To apply formula $s = r\theta$, it will first be necessary to find value of $\angle$ POQ in radians.

Now, $\qquad \sin \angle \text{SOQ} = \dfrac{a}{r} = \dfrac{2}{2{\cdot}05} = 0{\cdot}9756$

$\therefore$ From sine table, $\angle$SOQ = 77° 18′.

Using Circular or Radian table, $77° 18' = 1{\cdot}3491$ radians.

$$\therefore \angle POQ = \theta = 2{\cdot}6982 \text{ radians}$$

$$\therefore \text{Length of arc.} \quad s = r\theta$$
$$= 2{\cdot}05 \text{ ins.} \times 2{\cdot}6982$$
$$= \mathit{5{\cdot}531\ ins.}$$

(c) *Find height of arc* 1 *in. from end of base* [XY].

$(OX)^2 = (ZX)^2 + (ZO)^2$ (where $OX = 2{\cdot}05$ ins. and $ZX = 1$ in.).

$$\therefore \quad 2{\cdot}05^2 = 1^2 + (ZO)^2$$
$$\therefore \quad (ZO)^2 = 2{\cdot}05^2 - 1^2$$
$$\therefore \quad ZO = \sqrt{3{\cdot}05 \times 1{\cdot}05} = 1{\cdot}789 \text{ ins.}$$
$$\therefore \quad RZ = RO - ZO$$
$$= 2{\cdot}05 \text{ ins.} - 1{\cdot}789 \text{ ins.} = 0{\cdot}261 \text{ in.}$$
$$\therefore \quad XY = ZS = RS - RZ$$
$$= 1{\cdot}6 \text{ ins.} - 0{\cdot}261 \text{ in.}$$
$$= \mathit{1{\cdot}339\ ins.}$$

(d) *Find area of segment* PRQS.

Area of segment

$$= \text{area of sector POQR} - \text{area of } \triangle POQ$$
$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}PQ \,.\, SO$$
$$= \tfrac{1}{2} \,.\, (2{\cdot}05)^2 \,.\, 2{\cdot}6982 - \tfrac{1}{2} \,.\, 4 \,.\, (2{\cdot}05 - 1{\cdot}6)$$
$$= 5{\cdot}671 - (2 \times 0{\cdot}45)$$
$$= 5{\cdot}671 - 0{\cdot}9$$
$$= \mathit{4{\cdot}771\ sq.\ ins.}$$

Example 2

A wheel of diameter 5 *ft. makes* 40 *revolutions per min. Find its angular velocity per sec., and the linear speed of a point on its rim.*

$$1 \text{ rev.} = 2\pi \text{ radians}$$
$$\therefore \quad 40 \text{ revs. per min.} = 40 \times 2\pi \text{ radians per min.}$$
$$\therefore \quad \omega = \frac{40 \times 2\pi}{60} \text{ radians per sec.}$$

$\therefore$ *Angular velocity* $(\omega) = 4{\cdot}189$ *radians per sec.*

If $\quad v =$ speed of point on rim

$$v = r\omega$$
$$= \tfrac{5}{2} \times 4{\cdot}189$$
$$= 10{\cdot}47 \textit{ ft. per sec.}$$

Example 3

A motor car, racing on a circular track, covers an arc subtending 0·08 *radian at the centre every second. The radius of the track is* 500 *yds. How long does it take to travel a mile, and what is its speed in miles per hour?*

$$\omega = 0{\cdot}08 \text{ radian per sec.}$$

$$\therefore \quad v \text{ (linear speed)} = r\omega$$
$$= (500 \times 0{\cdot}08) \text{ yds. per sec.}$$
$$= 40 \text{ yds. per sec.}$$

$\therefore$ Time to travel 1 mile,

$$= \tfrac{1760}{40} \text{ sec.} = 44 \textit{ secs.}$$

$\therefore$ Speed in mls. per hour

$$= 40 \text{ yds. per sec.}$$
$$= 40 \times 60 \times 60 \text{ yds. per hr.}$$
$$= \frac{40 \times 60 \times 60}{1760} \text{ mls. per hr.}$$
$$= 81\tfrac{9}{11} \textit{ mls. per hr.}$$

Example 4

A circle of 4-*in. radius rolls outside a fixed circle of radius* 6 *ins. When the line joining the centres has turned through* 25°, *find the angle through which the rolling circle has turned.*

Let centre of rolling circle be initially at O^1 and finally at N (Fig. 33).

Then $\quad \angle NOO^1 = 25°$.

Then $\quad$ arc $C^1C = s = r\theta$ (in larger circle)

$$\therefore \quad C^1C = 6 \text{ ins.} \times 25° \text{ in radians}$$
$$= 6 \text{ ins.} \times 0{\cdot}4363.$$

Now, suppose the rolling circle rolls back again. Then C^1D will roll back on to C^1C, and if $C^1D = C^1C$, D will coincide with C.

Thus if $C^1D = s$, DN was originally on CO^1.

$\therefore$ The angle through which the rotating circle has rotated is $\angle NPO^1$, *i.e.*, the ext. angle of ΔONP. But ext. $\angle NPO^1 = \angle NOP + \angle ONP$

and $\angle NOP = 25°$

$\therefore$ It is necessary to find $\angle ONP$.

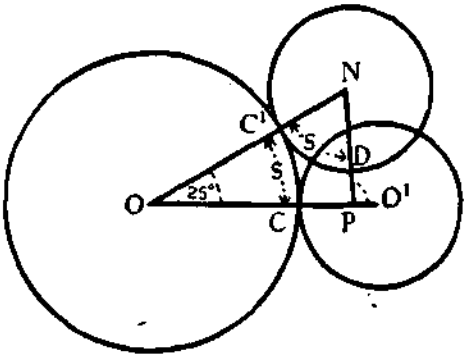

Fig. 33.

As $C^1D = s =$ arc subtending $\angle ONP$ at centre N

$$s = r\theta \text{ (where } \theta = \angle C^1ND)$$

But $s = 6 \text{ ins.} \times 0{\cdot}4363$ and $r = 4$ ins.

$$\therefore \quad 6 \times 0{\cdot}4363 = 4 \times \theta \text{ (in radians)}$$

$$\therefore \quad \theta = \frac{6 \times 0{\cdot}4363}{4} \text{ radians}$$

$$\therefore \quad \theta = 0{\cdot}6545 \text{ radian}$$

$$\therefore \quad \theta = 37\tfrac{1}{2}° = \angle ONP$$

$\therefore$ Angle through which radius CO^1 (*i.e.*, through which rotating circle has moved)

$$= 25° + 37\tfrac{1}{2}°$$
$$= 62\tfrac{1}{2}°$$

EXERCISE 9

1. A chord of length $2\frac{1}{2}$ ins. is drawn in a circle of radius 2 ins. Find the maximum heights of the two arcs cut off.

2. In a circle of diameter $3\frac{1}{2}$ ins. is drawn a chord of length 3 ins. Find the angle subtended at the centre of the circle and the lengths of the two arcs into which the circumference is divided.

3. A circular arc has a base of 5 yds. and a maximum height of 1 yd. What is its height at a distance of 2 ft. 3 ins. from its end, and at what distance from the end is it 1 ft. 6 in. high?

4. A chord of 3 ins. stands in a circle of 4 ins. diameter. Find the area of the sector and also of the segment.

5. Find the weight of the piece of iron shown (Fig. 34), if its thickness is 2 mms. and the density of iron 7·9 gms. per c.c.

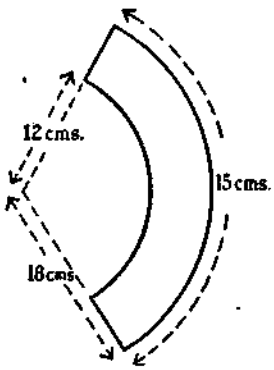

FIG. 34.

6. A circle of diameter 2·9 ins. rolls, without slipping, on the circumference of another (fixed) circle of diameter 5·2 ins. What angle is swept out by the line joining their centres, during one complete revolution of the rolling circle?

7. If a wheel is making 300 revs. per min., find its angular velocity per sec. If its diameter is 4 ins., find the speed of a point on its rim in ft. per sec.

8. A wheel 1 ft. 6 ins. in diameter is revolving with an angular velocity of 6 radians per sec. Find the linear speed of its rim in ft. per sec., and the number of revolutions the wheel makes per minute.

9. State the relation between the products of the segments of each of two intersecting chords of a circle. A loaded

beam rests on two supports distant L apart. Assuming that the beam is bent into an arc of a circle of radius R, and the deflection of the beam midway between the supports is d, show that

$$R = \frac{L^2}{8d} + \frac{d}{2}.$$

Obtain an expression for R when d is so small that d^2 may be neglected. (U.L.C.I., 1936.)

10. The radii AB, AC, 18 ft. long, of a circle include an angle of 45°. A concentric arc DE of radius 12 ft. cuts AB in D and AC in E. The figure BCED represents a plate of wrought iron $\frac{1}{4}$ in. thick, which is to be bent round to form a conical flue.

Calculate—

(i) the area of the plate in square feet;

(ii) the diameter in ft. at the top and bottom of the flue;

(iii) the weight of the flue if 1 cub. in. of the material weighs 0·28 lb. (U.E.I., 1935.)

11. A rod OAB rotates about its end O at an angular speed of w radians per sec. The linear speeds of A and B are p ft. per sec. and q ft. per sec., respectively, p being less than q. The length AB is c ins. Express w in terms of p, q and c. Find the value of w when $p = 5$, $q = 7$ and $c = 8$. (N.C.T.E.C., 1935.)

12. How is an angle measured in circular measure? A circular curve is a quarter of a mile long between the points A and B. The radius of the curve is 600 yds. Find the angle between the radius at A and that at B. Find also the angle between the tangent to the curve at A and that at B. Give answers in units of circular measure and also in degrees. (U.L.C.I., 1927.)

13. Using 3·18 as an approximation for $\frac{10}{\pi}$, find the angular speed in revolutions per min. of a fly-wheel rotating

at $23 \cdot 38$ radians per sec., and find the greatest error to which the answer is liable if each of the numbers $3 \cdot 18$ and $23 \cdot 38$ is liable to an error of $\pm 0 \cdot 005$. (N.C.T.E.C., 1931.)

14. A wheel, 4 ft. radius, rolls a distance of 9 ft. along level ground. Calculate the horizontal displacement of the point on the rim initially in contact with the ground and the height above the ground at the end of the movement.

(N.C.T.E.C., 1933.)

CHAPTER 6

MENSURATION (*continued*)

1. Prisms, the Cylinder

A prism is a solid generated by a straight line moving parallel to itself, while its end moves round the perimeter of a plane figure: it is thus a solid with two plane ends of the same size and shape.

A cylinder is a circular prism in which the straight line moves round the circumference of a circle.

If the moving line is at right angles to the plane figure (that is, if the length of the solid is at right angles to its ends), the prism is a right prism: if not, it is oblique.

We have previously learnt that for a right prism:

(1) *The area of the lateral surface = perimeter of base × height.*

(2) *The volume = area of base × height.*

In the special case of the *cylinder*.

$$\textit{Area of lateral surface} = 2\pi rh$$

$$\textit{Volume} = \pi r^2 h = \frac{\pi d^2 h}{4}$$

(where r, d and h are radius, diameter and height respectively).

For the *total* surface of cylinder, the two end-areas must be added, *i.e.*, $2\pi r^2$.

$$\therefore \quad \textit{Total surface of cylinder} = 2\pi rh + 2\pi r^2$$

$$\quad ,, \quad ,, \quad ,, \quad = 2\pi r(h + r).$$

2. Oblique Prism

If a pile of rectangular slabs, or of round discs be set up vertically to form a rectangular prism or a cylinder, and

then displaced slightly out of the vertical to form a "leaning tower," as in Figs. 35 and 36, it will be seen that the height, base-area and volume all remain the same as at first. Thus the volume is the product of the base-area and the height in

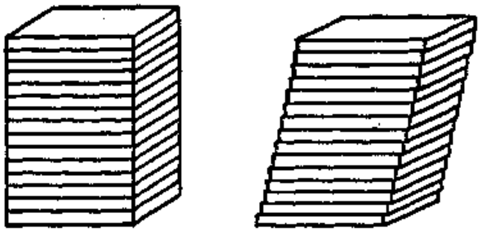

FIG. 35.

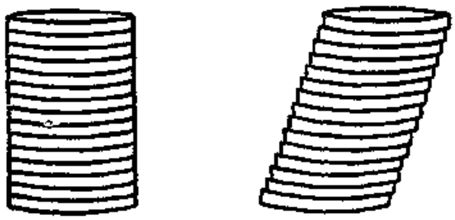

FIG. 36.

the case of an oblique prism (the height being the perpendicular height).

It will be noted, however, that the area of the lateral surface is not the same as before, since the height at right angles to the base edge is different on some of the faces. Those face-areas must be dealt with separately.

3. Frustum of Prism (and Cylinder)

When a plane section is taken through a prism parallel to its ends (*i.e.*, perpendicular to its length) the section is known as the "cross-section" of the prism, and the portion of the prism left is still a prism.

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If, however, the section be not parallel to the ends, the portion of the prism between the plane of section and the base is called a "frustum."

In Fig. 37, ABCD represents an oblique plane of section of the prism and the portion ABCDKLMN is a frustum.

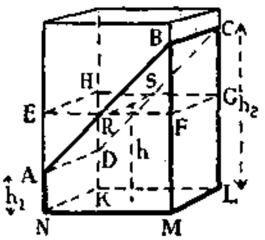

FIG. 37.

If the plane ABCD be cut across halfway between A and B by a horizontal plane EFGH (*i.e.*, a cross-section), a wedge-shaped piece RFGSCB will be cut off, which, if turned over, would fit exactly into the space ADSREH.

$$\therefore \quad \text{Vol. of frustum} = \text{vol. of prism below EFGH}$$
$$= \text{base-area KLMN} \times \text{height } h$$
$$\text{where } h = \tfrac{1}{2}(\text{AN} + \text{BM})$$
$$= \tfrac{1}{2}(h_1 + h_2)$$

$\therefore$ *Volume of Frustum = Base-area × Average Height.*

The area of the section ABCD (Fig. 37) can be found as follows:

Let $\angle \text{BRF} = \theta$ (Fig. 38)

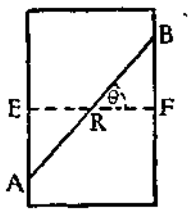

FIG. 38.

Then $\dfrac{\text{RF}}{\text{RB}} = \cos\theta$

$\therefore \quad \text{RF} = \text{RB}\cos\theta$

and $\text{RB} = \dfrac{\text{RF}}{\cos\theta}$

$\therefore \quad \text{AB} = \dfrac{\text{EF}}{\cos\theta}$

$$\text{Area of ABCD} = \text{AB} \times \text{BC (Fig. 37)}$$
$$= \frac{\text{EF}}{\cos\theta} \times \text{BC}$$

$$\text{But EF} \times \text{BC} = \text{EF} \times \text{FG}$$
$$= \text{area of cross-section of prism}$$

$\therefore$ *Area of section* ABCD $= \dfrac{\textit{area of cross-section of prism}}{\cos\theta}$.

This applies equally to the exposed face when an oblique planecuts through a cylinder.

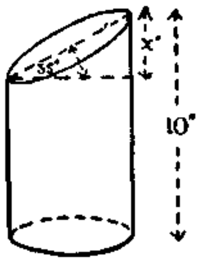

FIG. 39.

Example 1

(*a*) *The area of the cross-section of a cylinder is 52 sq. ins. What is the area of a section making an angle of 35° with the plane of the cross-section?*

$$\text{Area of oblique section} = \frac{\text{area of cross-section}}{\cos 35^\circ}$$

$$= \frac{52 \text{ sq. ins.}}{\cos 35^\circ}$$

$$= 63{\cdot}48 \textit{ sq. ins.}$$

(*b*) *What is the weight of the frustum if the greatest length is* 10 *ins. and the density of the material* 0·29 *lb. per cub. in.?*

Volume of frustum = area of end $\times$ average height.
To find average height :

$$\frac{x}{\text{diameter}} = \tan 35^\circ$$

But $\qquad \text{radius} = \sqrt{\frac{\text{Area}}{\pi}}$ (since $\pi r^2 = \text{A}$)

$$\therefore \quad \text{radius} = \sqrt{\frac{52}{\pi}}$$

$$= 4{\cdot}069 \text{ ins.}$$

$$\therefore \quad \text{diameter} = 2 \times 4{\cdot}069 \text{ ins.}$$

$$\therefore \quad x = \text{diameter} \times \tan 35^\circ$$

$$= 2 \times 4{\cdot}069 \times \tan 35^\circ$$

$$= 5{\cdot}698 \text{ ins.}$$

$$\therefore \quad \text{Average height} = 10 - \frac{5{\cdot}698}{2}$$
$$= 10 - 2{\cdot}849$$
$$= 7{\cdot}151 \text{ ins.}$$

$\therefore$ Volume of frustum $= 52 \times 7{\cdot}151$ cub. ins.

$\therefore$ Weight of frustum $= 52 \times 7{\cdot}151 \times 0{\cdot}29$ lb.
$= $ *107·8 lb.*

Example 2

Find the volume and the sum of the areas of the two curved surfaces of a hollow cylinder 6·5 *ins. long, if the internal and external diameters are* 2·2 *and* 3·8 *ins. respectively.*

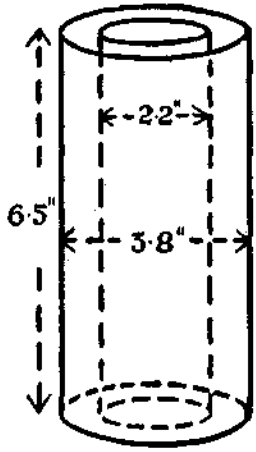

FIG. 40.

For volume,

$$\text{Vol.} = \text{area of end} \times \text{length}$$
$$= \left\{\frac{\pi D^2}{4} - \frac{\pi d^2}{4}\right\} \times l$$
$$= \frac{\pi l}{4}\{D^2 - d^2\}$$
$$= \frac{\pi l}{4}\{D + d\}\{D - d\}$$
$$= \frac{\pi \times 6{\cdot}5 \times 6 \times 1{\cdot}6}{4}$$
$$= \mathit{49{\cdot}0 \text{ cub. ins.}}$$

For sum of areas of curved surfaces.

Area of curved surface of cylinder
= perimeter of end $\times$ length.

$\therefore$ Total area of two surfaces

$$= \pi Dh + \pi dh$$
$$= \pi h(D + d)$$
$$= \pi \times 6{\cdot}5 \times 6$$
$$= \mathit{122{\cdot}6 \text{ sq. ins.}}$$

Example 3

A hollow shaft, whose inside diameter is 9 ins., weighs the same as a solid shaft of the same length and material, 5 ins. in diameter. What is the thickness of the material in the hollow shaft?

Let D = outside diameter of the hollow shaft.

Then, if weights of the two are equal (and of the same material), their volumes must be equal.

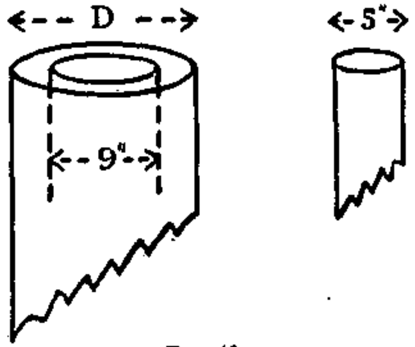

FIG. 41.

Volume of hollow shaft (per unit length) $= \frac{\pi}{4}(D^2 - 9^2)$

Volume of solid shaft (per unit length) $= \frac{\pi}{4}(5^2)$

$$\therefore \quad \frac{\pi}{4}(D^2 - 9^2) = \frac{\pi}{4}(5)^2$$

$$\therefore \quad D^2 - 9^2 = 5^2$$

$$\therefore \quad D^2 = 81 + 25 = 106$$

$$\therefore \quad D = \sqrt{106} = 10{\cdot}3 \text{ ins.}$$

$$\therefore \quad \text{Thickness of material} = \frac{10{\cdot}3 - 9}{2}$$

$$= \textit{0·65 in.}$$

4. Pyramids (including Cone)

If one end of a straight line of variable length be fixed, while the other end moves round the boundary of a plane figure, the solid generated is called a pyramid. The fixed end of the line is the vertex and the plane figure is the base.

If the line joining the vertex to the centre of the base is at right angles to the plane of the base, the pyramid is a right pyramid—if not, it is oblique.

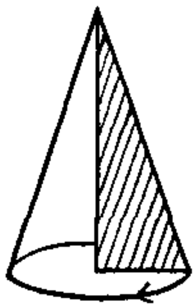

FIG. 42.

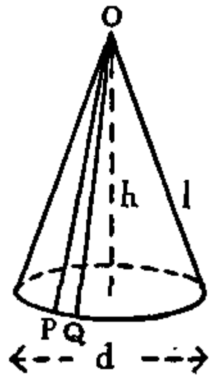

FIG. 43.

A pyramid with a circular base is a cone, the moving end of the line (in the above general definition) traversing the circumference of a circle; a right cone may also be looked upon as the solid generated by a right-angled triangle rotating round one of the sides containing the right angle (Fig. 42).

In the First-Year Course we have already seen that:

(1) the volume of a pyramid is one-third of the volume of the corresponding prism (*i.e.*, a prism with the same base-dimensions and height);

(2) the lateral surface of a pyramid is the product of half the perimeter of the base and the slant height.

These statements apply to the cone.

Let d = base-diameter, h = height, l = slant-height (Fig. 43).

Then $\quad Volume = \frac{1}{3}$ (base-area × height)

$$= \frac{1}{3} \cdot \frac{\pi d^2}{4} \cdot h$$

$$= \frac{\pi d^2 h}{12}$$

The area of the slant surface of the cone may be regarded as the sum of a very large number of exceedingly narrow triangular areas such as OPQ (by the method shown on p. 88).

But $\quad$ area of OPQ $= \dfrac{\text{base PQ} \times \text{height QO}}{2}$

∴ Sum of all such areas drawn to cover the conical surface

$$= \frac{\text{QO} \times (\text{PQ} + \ldots)}{2}$$

$$= \frac{\text{QO} \times \text{circumference of base}}{2}$$

$$= \frac{2\pi r \times l}{2}$$

∴ *Area of slant surface* $= \pi rl$.

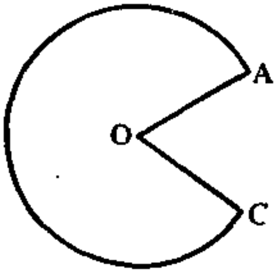

FIG. 44.

Alternatively, we can suppose a sector to be cut from a circle as AOC (Fig. 44) and the rest drawn together so that OC fits OA. Then the centre will "hump" up and so form a cone whose curved surface has an area equal to the area of the remaining part of the circle and whose slant height (l) will be the length of OA (r).

∴ Area of slant surface of cone

$$= \text{area of sector}$$

$$= \tfrac{1}{2}(\text{arc} \times \text{radius}) \text{ (see p. 89).}$$

But the arc becomes $2\pi r$ in the cone and the radius becomes l. (Fig. 45.)

$\therefore$ Area of slant surface $= \frac{1}{2} \cdot 2\pi r \cdot l$

$= \pi rl$ (*as before*)

(Note that $l^2 = h^2 + r^2$

$\therefore$ Above area may be written $\pi r\sqrt{h^2 + r^2}$).

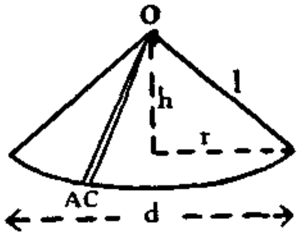

FIG. 45.

To find the total surface of the cone, add the area of the base.

$$\therefore \text{Total surface} = \pi r\sqrt{h^2 + r^2} + \pi r^2$$
$$= \pi r\{\sqrt{h^2 + r^2} + r\}$$
$$= \pi r\{l + r\}.$$

5. Frustum of Pyramid (and Cone)

If a pyramid be cut through by a plane *parallel* to its base, the portion of the pyramid between that plane and the base is called a " frustum " of a pyramid.

(Note that here the cutting plane is parallel to the base, whereas in the case of the prism the plane was *not* parallel to the base.)

Fig. 46 shows a frustum of a pyramid.

The area of the lateral surface, and the volume, can each be found by subtracting the surface area, or volume, of the part cut off, from the surface area, or volume, of the whole pyramid.

In some cases it is quicker to find the areas of the separate faces by applying the rule for the area of a trapezoid.

Area of trapezoid $= \frac{1}{2}$ sum of parallel sides $\times$ slant height perpendicular to those sides.

$\therefore$ *Total lateral surface of frustum of pyramid (when regular) $= \frac{1}{2}$ sum of perimeters of the two ends $\times$ slant height.*

$\therefore$ *Lateral surface of frustum of cone* $= \frac{1}{2}(2\pi r + 2\pi R) \times l$
$= \pi l(r + R)$

(*where* R *and* r *are radii of two ends of frustum and* l *is the slant height*).

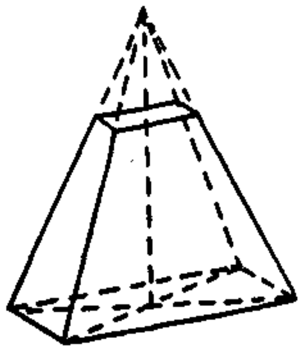

FIG. 46.

Since $$\pi l(R + r) = 2\pi\left(\frac{R + r}{2}\right)l$$

we may write:

Lateral surface of frustum of cone

$= 2\pi xl$ (where $x =$ average of the two radii)

$=$ lateral surface of a cylinder of radius x and height l.

$\therefore$ The curved surface of a frustum of a cone is equal to the curved surface of a cylinder whose height is the slant side of the frustum and whose base-radius is the average length of the radii of the ends of the frustum.

6. To Find the Volume of a Frustum of a Pyramid

Let the areas of the two ends be A and B respectively, the perpendicular height of the frustum, h, and the height of the portion cut away, x.

Then

Volume of whole pyramid $= \frac{1}{3} \cdot A \cdot (h + x)$ (by previous rule)

Volume of part cut off $= \frac{1}{3} \cdot Bx$

$$\begin{aligned}\therefore \text{ Volume of frustum} &= \tfrac{1}{3}A(h + x) - \tfrac{1}{3}Bx \\ &= \tfrac{1}{3}Ah + \tfrac{1}{3}Ax - \tfrac{1}{3}Bx \\ &= \tfrac{1}{3}Ah + \tfrac{1}{3}x(A - B).\end{aligned}$$

But by similar figures,

$$\left(\frac{x}{h+x}\right)^2 = \frac{B}{A} \text{ (since } x \text{ and } h \text{ are lengths and A and B are areas)}$$

$$\therefore \quad \frac{x}{h+x} = \frac{\sqrt{B}}{\sqrt{A}}$$

Cross multiplying,

$$\therefore \quad x\sqrt{A} = (h+x)\sqrt{B}$$

$$\therefore \quad x\sqrt{A} = h\sqrt{B} + x\sqrt{B}$$

$$\therefore \quad x(\sqrt{A} - \sqrt{B}) = h\sqrt{B}$$

$$\therefore \quad x = \frac{h\sqrt{B}}{\sqrt{A} - \sqrt{B}}.$$

Substituting this value for x in above,
Volume of frustum

$$= \tfrac{1}{3}Ah + \tfrac{1}{3}\left(\frac{h\sqrt{B}}{\sqrt{A} - \sqrt{B}}\right) . (A - B)$$

$$= \tfrac{1}{3}Ah + \tfrac{1}{3}(h\sqrt{B})(\sqrt{A} + \sqrt{B}) \text{ (cancelling out } \sqrt{A} - \sqrt{B})$$

$$= \tfrac{1}{3}h\{A + B + \sqrt{AB}\}.$$

Applying the formula to the volume of a conical frustum :
Volume of frustum of cone

$$= \tfrac{1}{3}h\{\pi R^2 + \pi r^2 + \sqrt{\pi^2 R^2 r^2}\}$$

(where R and r = radii of ends and h = height of frustum)

$$= \tfrac{1}{3}h\{\pi R^2 + \pi r^2 + \pi R r\}$$

$$= \tfrac{1}{3}\pi h\{R^2 + r^2 + Rr\}.$$

Example 1

A conical cover is required for a chimney. It must be 12 ins. in diameter at the base and 8 ins. high. What amount of sheet metal will it use and what size and shape must be cut out to make it (neglecting overlapping)?

From Fig. 47 it is seen that if l be the slant side

$$l = \sqrt{6^2 + 8^2}$$

$$\therefore \quad l = 10 \text{ ins.}$$

$\therefore$ Area of sheet metal needed

$$= \pi r l$$
$$= \pi \times 6 \times 10$$
$$= 60\pi \text{ sq. ins.}$$
$$= \mathit{188{\cdot}5\ sq.\ ins.}$$

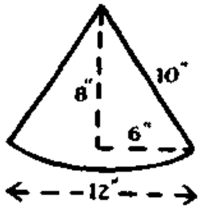

FIG. 47.

For sector.

As the circumference of the base $= 12\pi$ ins.

$\therefore$ length of arc of sector $= 12\pi$ ins. $(= s)$

and radius of sector $= 10$ ins. (slant side of cone).

Since $s = r\theta$ (where θ is in radians)

$$\therefore \quad 12\pi = 10\theta(\text{radians})$$

$$\therefore \quad \theta(\text{radians}) = \frac{12\pi}{10}$$

$$= \frac{6 \times 360^\circ}{10} \text{ (since } 2\pi \text{ radians} = 360^\circ)$$

$$= 216^\circ$$

$\therefore$ *From a circle of sheet metal of radius* 10 *ins., a sector whose angle at the centre is* 144° *must be cut and the remainder is the shape and size required.*

Example 2

Two pails are each 10 *ins. in depth. One is of uniform width,* 8 *ins., while the other is in the shape of a conical frustum,* 7 *ins. wide at one end and* 9 *ins. at the other. Which holds the more liquid and how much more?*

Vol. of cylindrical pail $= \dfrac{\pi d^2 h}{4}$

$$= \frac{\pi \times 8^2 \times 10}{4}$$

$$= \mathit{160\pi\ cub.\ ins.}$$

Vol. of conical pail

$$= \frac{\pi h}{3}\{R^2 + Rr + r^2\}, \text{ where R and } r \text{ are radii}$$

$$= \frac{\pi \times 10}{3}\{(\tfrac{7}{2})^2 + (\tfrac{7}{2} \cdot \tfrac{9}{2}) + (\tfrac{9}{2})^2\}$$

$$= \frac{\pi \times 10}{3}\left\{\frac{49 + 63 + 81}{4}\right\}$$

$$= \frac{\pi \times 10 \times 193}{4 \times 3}$$

$$= \pi \times 10 \times 16{\cdot}0833$$

$$= \mathit{160{\cdot}833\pi \text{ cub. ins.}}$$

$\therefore$ The conical pail holds $0{\cdot}833\pi$ cub. in. more

$\therefore$ The conical pail holds *2·618 cub. ins. more.*

7. The Sphere

A sphere is the solid generated by a semi-circle revolving about its diameter as axis.

The section of a sphere made by a plane is a circle: if the plane passes through the centre of the sphere, the circle is known as a "great circle"; other sections, made by planes not passing through the centre, are "small circles."

The various formulæ dealing with the surface-area, volume, etc., of a sphere are difficult to prove at this stage of the student's work; as they can be proved so much more easily by the methods of the Calculus, to be dealt with later (Book 3), we shall merely state those which the student will find useful.

8. Area of Surface of Sphere

If A is the surface-area of a sphere of radius, r,

$$A = 4\pi r^2$$

This area is equal to that of four great circles cutting through the sphere.

9. Volume of Sphere

If V is the volume of a sphere of radius, r,

$$V = \tfrac{4}{3}\pi r^3$$

A form often used in calculation by logarithms is

$$V = \frac{\pi d^3}{6}$$

where d is the diameter.

10. Relation between Volumes of Cylinder, Sphere and Cone

An interesting relation between these volumes is found, if a cylinder, sphere and cone be taken of equal diameters and heights as shown in Fig. 48.

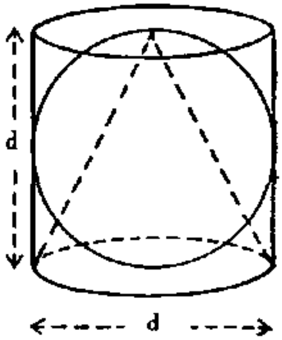

FIG. 48.

$$\text{Vol. of cylinder} = \frac{\pi d^2 h}{4} = \frac{\pi d^3}{4} \text{ (since } d = h\text{)}$$

$$\text{Vol. of sphere} = \frac{\pi d^3}{6}$$

$$\text{Vol. of cone} = \tfrac{1}{3}\,\frac{\pi d^2 h}{4} = \frac{\pi d^3}{12}$$

$$\text{But } \frac{\pi d^3}{4} : \frac{\pi d^3}{6} : \frac{\pi d^3}{12} = 3 : 2 : 1.$$

Thus the respective volumes of the cylinder, sphere and cone of equal diameters and heights are in the ratio 3 : 2 : 1.

Example

What is the thickness of a spherical shell (i.e., a hollow sphere) if its inner diameter is 1 ft., and its weight the same as that of a solid sphere of the same density and of diameter 11 ins.?

Since the weights and densities are equal, the volumes of actual material must be equal.

$$\text{Volume of solid sphere} = \frac{\pi d^3}{6}$$

$$= \frac{\pi \times 11^3}{6} \text{ cub. ins.}$$

$$= \frac{\pi \times 1331}{6} \text{ cub. ins.}$$

$$\text{Volume of material of hollow sphere} = \frac{\pi D^3}{6} - \frac{\pi d^3}{6}$$

(where D and d are the outer and inner diameters respectively).

$$\therefore \quad \text{Volume of hollow sphere} = \frac{\pi}{6}\{D^3 - 12^3\} \text{ cub. ins.}$$

$$= \frac{\pi}{6}\{D^3 - 1728\} \text{ cub. ins.}$$

$\therefore$ For equal volumes,

$$\frac{\pi}{6}\{D^3 - 1728\} = \frac{\pi}{6} \cdot 1331$$

$$\therefore \quad D^3 - 1728 = 1331$$

$$\therefore \quad D^3 = 3059$$

$$\therefore \quad D = 14{\cdot}52 \text{ ins.}$$

$\therefore$ Thickness of material of shell

$$= \frac{14{\cdot}52 - 12}{2}$$

$$= \mathit{1{\cdot}26\ ins.}$$

EXERCISE 10

1. A cylindrical column 4 ft. wide and 6 ft. high is surmounted by a cone of the same width and 3 ft. high. Find the amount of sheet-metal required to cover its whole lateral surface.

2. A 6-ft. length of metal pipe of outside diameter $4\frac{1}{2}$ ins. weighs 40 lb. If the density of the metal is 0·28 lb. per cub. in., find the thickness of the metal.

3. A pyramid whose base is 1 ft. square and whose height is 8 ins. is cut through by a horizontal plane 3 ins. from the apex. Find the volume of the frustum remaining and also the total area of its slant sides.

4. A cone 12 ins. high is cut across 5 ins. from the vertex. The volume of the frustum left is 200 cub. ins. Find the diameter of the base of the cone.

5. Find the weight of a conical flue whose top and bottom diameters are 1 ft. and 2 ft. respectively. The height of the flue is 2 ft. 6 ins. and the sheet metal of which it is made weighs $2\frac{1}{2}$ lb. per sq. ft.

6. A hollow iron sphere weighs 3 lb. Its outside diameter is 5 ins. Find its inside diameter if the density of iron is 450 lb. per cub. ft.

7. A conical hot-water can is to hold 1 gal. Its top and bottom are to be 5 ins. and 7 ins. in diameter respectively. Find its height (1 pint = 34·66 cub. ins.).

8. A horizontal cylindrical tank 5 ft. in diameter and 10 ft. long is partly filled with liquid, the greatest depth of which is 2 ft. How many gallons are there in the tank? (1 gal. = 0·16 cub. ft.).

9. The ends of the frustum of a cone cover 113·1 sq. cms. and 314·2 sq. cms. respectively. The area of the curved surface is equal to that of a sphere of diameter 7·6 cms. Find the slant side of the frustum.

10. A vessel whose shape is that of a cone with vertex downwards contains liquid to a depth of 2 ins. Upon adding 45 cub. ins. of liquid, the depth becomes 5 ins. By using the properties of similar solids, find what further volume of liquid must be added in order that the depth of the liquid may become 6 ins. (N.C.T.E.C., 1935.)

11. A solid pyramid, whose total surface area is three times the area of the base, is cut into two parts by a plane parallel to its base, midway between its vertex and base. The area of the cross-section of the pyramid by the plane is 5 sq. ins. Find the total surface area of each of the two parts into which the pyramid is divided by the plane.

(N.C.T.E.C., 1934.)

CHAPTER 7

PROGRESSIONS

1. Meaning of a Series

A sequence of quantities, connected by a definite law, is called a " series "; each of the quantities is called a term of the series; the law connecting the terms is called the " law of formation."

Simple examples of numbers " in series " are:

(1) 1, 2, 3, 4, 5, . . .
(2) 1, 3, 5, 7, 9, . . .
(3) 2, 4, 6, 8, 10, ...
(4) 1, 4, 9, 16, 25, . . . (squares of the natural numbers).
(5) 1, 8, 27, 64, 125, . . . (cubes of the natural numbers).
(6) 1, 5, 9, 13, 17 . . . (add 4 to each in succession).
(7) 5, 10, 20, 40, 80 . . . (multiply by 2).
(8) 4, 5·3, 6·6, 7·9, . . . (add 1·3).
(9) 1, $-$ 1, 1, $-$ 1, . . . (multiply by $-$ 1).
(10) 100, $-$50, 25, $-$ 12·5 . . . (multiply by $-$ 0·5).

2. Arithmetic Progressions

An Arithmetic Progression is a series in which each term is formed from the preceding one by the addition (or subtraction) of the same quantity. Thus the terms increase (or decrease) by a fixed amount, called the *common difference.*

3. Examples of Arithmetic Series

(1) 1, 5, 9, 13, 17, . . . (common difference is $+$ 4).
(2) 16, 10, 4, $-$ 2, $-$ 8, . . . (common difference is $-$ 6).

(3) $x, x + y, x + 2y, x + 3y, \ldots$ (common difference is $+y$).
(4) Yearly salaries rising by fixed annual increments.
(5) Successive yearly values of a car, if it depreciates by a constant amount annually.
(6) The heights of the supports of a sloping hand-rail, when equally spaced out and resting on a level base.

4. General Expression for the Terms in an A.P.

Let the first term be a, and the common difference d. Then the series can be expressed by

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

By noting that

the first term is a
the second term is $a + (2 - 1)d$
the third term is $a + (3 - 1)d$
the fourth term is $a + (4 - 1)d$

we see that the

twelfth term would be $a + (12 - 1)d = a + 11d$
twentieth term would be $a + (20 - 1)d = a + 19d$

and thus any term, the nth, could be written

$$\textbf{nth term} = a + (n - 1)d$$

Examples

(1) *Write down the tenth and the twentieth terms of* 1, 6, 11 . . .

The nth term $= a + (n - 1)d$
where $a = 1, d = 5$

$$\therefore \text{ tenth term} = a + 9d$$
$$= 1 + (9 \times 5)$$
$$= 46$$

$$\text{twentieth term} = a + 19d$$
$$= 1 + (19 \times 5)$$
$$= 96$$

(2) *If the first term of an A.P. is 5, and the seventh is 29, find the eighteenth term.*

$$\text{first term} = a = 5$$
$$\text{seventh term} = a + 6d = 29$$

$\therefore$ by subtraction, $6d = 24$

$$\therefore \quad d = 4$$

But eighteenth term $= a + 17d$

$$\therefore \quad \text{eighteenth term} = 5 + (17 \times 4)$$
$$= 73$$

(3) *Which term of the series* 2·3, 4·2, 6·1 . . . *is* 36·5?

Let the nth term $= 36{\cdot}5$

$$\therefore \quad a + (n-1)d = 36{\cdot}5$$

But $a = 2{\cdot}3$ and $d = 1{\cdot}9$

$$\therefore \quad 2{\cdot}3 + (n-1)1{\cdot}9 = 36{\cdot}5$$
$$\therefore \quad 2{\cdot}3 + 1{\cdot}9n - 1{\cdot}9 = 36{\cdot}5$$
$$\therefore \quad 1{\cdot}9n = 36{\cdot}1$$
$$\therefore \quad n = 19$$

$\therefore$ 36·5 is the *nineteenth term of the series*

(4) *Insert six Arithmetic Means between* 10 *and* 25·4.

This means that we must put six numbers between 10 and 25·4, such that 10, ?, ?, ?, ?, ?, ?, 25·4 form an Arithmetic Progression.

$\therefore$ 25·4 must be the eighth term and 10, the first.

$$\therefore \quad \text{eighth term} = a + 7d = 25{\cdot}4$$

and

$$\text{first term} = a = 10$$

$\therefore$ By subtraction,

$$7d = 15{\cdot}4$$
$$\therefore \quad d = 2{\cdot}2$$

$\therefore$ The series will be

10, 12·2, 14·4, 16·6, 18·8, 21, 23·2, 25·4

and the required Arithmetic Means will be

12·2, 14·4, 16·6, 18·8, 21, and 23·2

(5) *A motor car is bought for £500. If its value depreciates at the rate of £30 per year, what is its value at the end of 9 years?*

At the end of: 1 yr. 2 yrs. 3 yrs. . . . 9 yrs.

value: 470 440 410 . . . ?

It is plain that we require the ninth term of the series 470, 440, 410 . . .

where $a = 470$ and $d = -30$

$$\begin{aligned} \text{ninth term} &= a + 8d \\ &= 470 - (8 \times 30) \\ &= 470 - 240 \\ &= \pounds 230 \end{aligned}$$

5. To Find the Sum of n Terms of an A.P.

Let S_n denote the sum of n terms of a series in A.P., whose first term $= a$, and whose common difference $= d$.

Then

$$S_n = a + (a + d) + (a + 2d) + \ldots + \{a + (n - 1)d\}$$

Now let l denote the nth or last term, $a + (n - 1)d$

Then

$$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$$

By writing the series in reverse order,

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$$

By adding these two series together,

$$\begin{aligned} \therefore\ 2S_n &= (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l) \\ &= (a + l) \ldots \text{ to } n \text{ terms} \\ &= n(a + l) \end{aligned}$$

$$\therefore\ S_n = \frac{n}{2}\{a + l\}$$

But $l = a + (n - 1)d$

$$\therefore \quad S_n = \frac{n}{2}\{a + a + (n - 1)d\}$$

$$\therefore \quad S_n = \frac{n}{2}\{2a + (n - 1)d\}$$

Examples

(1) *Find the sum of* $3{\cdot}2 + 4{\cdot}1 + 5{\cdot}0 \ldots$ *to ten terms*

$$a = 3{\cdot}2,\ d = 0{\cdot}9 \text{ and } n = 10$$

$$\begin{aligned}\therefore \quad S_{10} &= \tfrac{10}{2}\{6{\cdot}4 + (9 \times 0{\cdot}9)\}\\ &= 5\{6{\cdot}4 + 8{\cdot}1\}\\ &= 5 \times 14{\cdot}5\end{aligned}$$

$$\therefore \quad S_{10} = 72{\cdot}5$$

(2) *How many terms of the series* $2\tfrac{1}{2}, 5, 7\tfrac{1}{2}, 10 \ldots$ *will give a sum of* $137\tfrac{1}{2}$?

Here $a = 2\frac{1}{2}$, $d = 2\frac{1}{2}$, and $S_n = 137\frac{1}{2}$.

It is required to find n.

$$\begin{aligned}\therefore \quad 137\tfrac{1}{2} &= \frac{n}{2}\{5 + (n - 1)2\tfrac{1}{2}\}\\ &= \frac{n}{2}\{2\tfrac{1}{2} + 2\tfrac{1}{2}n\}\\ \therefore \quad 275 &= n\{2\tfrac{1}{2} + 2\tfrac{1}{2}n\}\\ &= 2\tfrac{1}{2}n + 2\tfrac{1}{2}n^2\end{aligned}$$

Dividing both sides by $2\frac{1}{2}$

$$110 = n + n^2$$

$\therefore$ Solving for n

$$n^2 + n - 110 = 0$$

$$(n + 11)(n - 10) = 0$$

$$\therefore \quad n = -11 \text{ or } 10$$

As $n = -11$ is evidently meaningless in this case, $n = 10$.

$\therefore$ *The number of terms to give a sum of* $137\frac{1}{2}$ *is* 10.

(3) *Find the number of terms in an A.P., whose third term is 2, whose sixth term $= -13$ and whose sum $= -105$.*

$$S_n = -105$$
$$\text{Sixth term} = a + 5d = -13$$
$$\text{Third term} = a + 2d = 2$$

Subtracting $\therefore\ 3d = -15$

$$\therefore\ d = -5$$

Substituting $d = -5$ in $a + 2d = 2$

$$a - 10 = 2$$
$$\therefore\ a = 12$$

Now substituting $a = 12$, $d = -5$, $S_n = -105$, in the formula

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

we obtain,

$$-105 = \frac{n}{2}\{24 + (n-1)(-5)\}$$
$$= \frac{n}{2}\{24 - 5n + 5\}$$
$$\therefore\ -210 = 29n - 5n^2$$
$$\therefore\ 5n^2 - 29n - 210 = 0$$
$$\therefore\ (5n + 21)(n - 10) = 0$$
$$\therefore\ n = -\tfrac{21}{5} \text{ or } 10.$$

Rejecting the negative value as meaningless in this case, we have,

Number of terms required is 10.

(4) *From a piece of wire 8 ft. long, a short piece is cut, then a piece $\frac{1}{5}$ in. longer than the first, then another $\frac{1}{5}$ in. longer than the second, and so on. If 30 pieces so cut exactly use up the wire, find the length of the first piece.*

Let the length of the first piece $= a$ ins.

Then the lengths of the other pieces are

$$a + \tfrac{1}{5} \text{ ins.}, a + \tfrac{2}{5} \text{ ins.} \ldots$$

$\therefore$ If the total length of 30 pieces $= S_n$

$$S_n = \tfrac{30}{2}\{2a + (29 \times \tfrac{1}{5})\}$$
$$\therefore \quad 8 \text{ ft.} = 96 \text{ ins.} = 15\{2a + \tfrac{29}{5}\}$$
$$\therefore \quad 96 = 30a + 87$$
$$\therefore \quad 30a = 9$$
$$\therefore \quad a = \tfrac{9}{30} = 0{\cdot}3 \text{ ins.}$$

$\therefore$ *Length of first piece = 0·3 ins.*

EXERCISE 11

1. Find the ninth term of 17, 13, 9 . . .
2. Find the eleventh term of -7, $-5{\cdot}5$, -4, . . .
3. Find the twenty-fifth term of 0·6, 0·72, 0·84 . . .
4. Find the twelfth term of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$ . . .
5. Find the nth term of 2, 7, 12 . . .
6. Find the pth term of 6, 4, 2, . . .
7. The tenth term of a series in A.P. is 18 and the fourteenth is 30. Find the fifth term.
8. The fifth term of an A.P., is 15, and the twelfth is 43. Find the ninth term.
9. Find the tenth term of an A.P. whose third term is 2·9 and whose seventh term is 1·3.
10. Which term of the series $7\frac{1}{4}$, 10, $12\frac{3}{4}$ . . . is $37\frac{1}{2}$?
11. Which term of the series $\frac{1}{3}$, $\frac{5}{9}$, $\frac{7}{9}$, . . . is $2\frac{5}{9}$?
12. Find the series which has ten Arithmetic Means between 20 and -20.
13. Find the missing terms in ?, ?, ?, 12, ?, ?, 30.
14. Insert four Arithmetic Means between 0·16 and 0·31.
15. Insert three Arithmetic Means between x and y.
16. If the first term of an A.P. is 21, the last 193, and the common difference is 2, find the number of terms.
17. Sum 13, 10, 7 . . . to 10 terms.
18. Sum 9·6, 7·3, 5·0 . . . to 14 terms.

19. Find the sum of twelve terms of $-6, -2, +2 \dots$

20. Sum the first ten terms of a series in A.P. if the first and last terms are 3 and 39 respectively.

21. How many terms of the series 11, 15, 19, etc., will give a sum of 341?

22. If a body falls 16 ft. in the first second of its flight, 48 ft. in the second second., 80 ft. in the third second, and so on, how far does it fall during the twentieth second, and how far has it fallen altogether in the 20 secs.? In how many seconds will it have fallen 4096 ft.?

23. A parent puts away a crown (5*s.*) on his son's first birthday, 2 crowns on his second birthday, 3 on his third, and so on. How much will the boy have when he is 10 yrs. old and how old must he be before the total is £34?

24. A man starts in business and loses £150 in the first year, £120 in the second year, and £90 in the third year. If the improvement continues at the same rate, find his total profit or loss at the end of 20 yrs. When would his losses be just balanced by his gains?

25. A contractor agrees to sink a well 250 ft. deep at a cost of 2*s.* 9*d.* for the first foot, 2*s.* 10½*d.* for the second foot and an extra 1½*d.* for each additional foot. Find the total cost and also the cost of the last foot.

26. A clerk in one office commences at £100 a year and receives a yearly rise of £10. Another clerk in another office commences at £100, but receives a rise of £44 after every fourth year. Which receives the greater total salary and by how much in (*a*) 32 yrs., (*b*) 36 yrs.?

27. A boy builds his bricks into a wall in such a way that each row contains one brick less than the row below it. How many rows can he build with 200 bricks and how many will be left over? (The top row contains only 1 brick.)

28. A cyclist's average speed per hour diminishes by ½ m.p.h. during each hour that he rides. If he covers 172½ miles in 10 hrs., find his average speed during the first hour.

29. Find the sum of fourteen terms of the A.P. whose first term is 11 and common difference 9.

(U.L.C.I., 1935.)

30. Find the thirtieth term and the sum of thirty terms of the series 4, 8, 12, 16. (U.L.C.I., 1936.)

6. Geometric Series

A series of terms each of which is formed by multiplying the term which precedes it by a constant factor is called a Geometric Series or Geometric Progression. The constant factor is called the *Common Ratio* of the series.

Examples

(1) 4, 8, 16, 32, . . . are in G.P. (common ratio is 2).
(2) $1, \frac{1}{4}, \frac{1}{16}, \ldots$ are in G.P. (common ratio is $\frac{1}{4}$).
(3) $9, -27, +81, \ldots$ are in G.P. (common ratio is -3).
(4) $x, rx, r^2x \ldots$ are in G.P. (common ratio is r).
(5) If a ball bounces up to a height equal to $\frac{5}{6}$ of the height from which it fell, the successive heights to which it rises form a G.P. If h is the initial height from which it drops, the series will be

$$h, \tfrac{5}{6}h, (\tfrac{5}{6})^2h, (\tfrac{5}{6})^3h, \text{ etc.}$$

7. General Expression for a Series in G.P.

Let the first term be denoted by a, and the common ratio by r.

Then the series can be expressed by

$$a, ar, ar^2, ar^3, \ldots$$

The relation between the index of r and the number of the corresponding term should be carefully noted:

$$\text{first term} = a$$
$$\text{second term} = ar \text{ (index of } r = 1)$$
$$\text{third term} = ar^2 \text{ (index of } r = 2)$$

seventh term $= ar^6$ (index of $r = 6$)
tenth term $= ar^9$ (index of $r = 9$)
nth term $= ar^{n-1}$ (index of $r = n - 1$)

It will be seen that the index of r is always *one* less than the number of the term in the series. We can now write down any required term in a given series, viz.:

$$\mathbf{n\text{th term} = ar^{n-1}}$$

Examples

(1) *Write down the sixth term of* 6, 18, 54, . . .
Here first term $a = 6$ and $r = 3$

$$\therefore \quad \text{sixth term} = ar^5$$
$$= 6 \times 3^5$$
$$= 6 \times 243$$
$$= \mathit{1458.}$$

(2) *Write down the seventh term of* $6, -4, +2\frac{2}{3}$ . . .
In this case $a = 6$

and $$r = \frac{-4}{6} = -\frac{2}{3}$$

$$\therefore \quad \text{seventh term} = ar^6$$
$$= 6 \times (-\tfrac{2}{3})^6$$
$$= 6 \times \tfrac{64}{729}$$
$$= \frac{128}{243}$$

(3) *Find the first five terms of a series in G.P. whose second term* $= 24$ *and whose sixth term* $= 121\frac{1}{2}$.

second term $= ar = 24$
sixth term $= ar^5 = 121\frac{1}{2}$

Dividing the sixth term by the second,

$$\frac{ar^5}{ar} = \frac{121\frac{1}{2}}{24}$$
$$\therefore \quad r^4 = \tfrac{243}{4} = \tfrac{81}{16}$$
$$\therefore \quad r = \sqrt[4]{\tfrac{81}{16}}$$
$$\therefore \quad r = \tfrac{3}{2}$$
$$\therefore \quad r = 1{\cdot}5$$

Now substitute 1·5 for r in $ar = 24$,

$$\therefore \quad a \times 1{\cdot}5 = 24$$

$$\therefore \quad a = \frac{24}{1{\cdot}5} = 16 = \text{first term}$$

$\therefore$ The first five terms of the series are

16, 24, 36, 54, 81

(4) *Insert four Geometric Means between* 100 *and* 10.

This is equivalent to putting four terms between 100 and 10 such that all the six numbers form a G.P.

$\therefore$ 100 ? ? ? ? 10 form a G.P.

$$\therefore \quad \text{sixth term} = ar^5 = 10$$
$$\text{first term} = a = 100$$

$$\therefore \quad \frac{ar^5}{a} = \tfrac{10}{100}$$

$$\therefore \quad r^5 = 0{\cdot}1$$

$$\therefore \quad r = \sqrt[5]{0{\cdot}1}$$

$$\therefore \quad r = 0{\cdot}6310$$

$$\begin{aligned} \therefore \quad \text{first term} &= 100 \\ \text{second term} &= 100 \times 0{\cdot}6310 = 63{\cdot}10 \\ \text{third term} &= 63{\cdot}10 \times 0{\cdot}6310 = 39{\cdot}81 \\ \text{fourth term} &= 39{\cdot}81 \times 0{\cdot}6310 = 25{\cdot}12 \\ \text{fifth term} &= 25{\cdot}12 \times 0{\cdot}6310 = 15{\cdot}85 \\ \text{sixth term} &= 10 \end{aligned}$$

$\therefore$ *The required Geometric Means are 63·10, 39·81, 25·12, 15·85.*

(5) *Which term of the series* 1·01, 0·909, 0·8181 . . . *is* 0·4827 ?

Let the required term be the nth term.

Then $a = 1{\cdot}01$ and $r = \dfrac{0{\cdot}909}{1{\cdot}01} = 0{\cdot}9$

and the nth term, $ar^{n-1} = 1{\cdot}01 \times (0{\cdot}9)^{n-1} = 0{\cdot}4827$

$$\therefore \quad (0{\cdot}9)^{n-1} = \frac{0{\cdot}4827}{1{\cdot}01}.$$

Taking logs. of both sides,

$$(n - 1) \log 0{\cdot}9 = \log 0{\cdot}4827 - \log 1{\cdot}01$$

$$\therefore \quad (n - 1) \times \bar{1}{\cdot}9542 = \bar{1}{\cdot}6836 - 0{\cdot}0043$$

$$\therefore \quad n - 1 = \frac{\bar{1}{\cdot}6836 - 0{\cdot}0043}{\bar{1}{\cdot}9542}$$

$$= \frac{\bar{1}{\cdot}6793}{\bar{1}{\cdot}9542}$$

$$= \frac{-0{\cdot}3207}{-0{\cdot}0458}$$

$$= \frac{0{\cdot}3207}{0{\cdot}0458}$$

$$= 7$$

$$\therefore \quad n = 8$$

$\therefore$ *0·4827 is the eighth term.*

8. To Find the Sum of *n* Terms of a G.P.

Let S_n denote the sum of n terms of a series in G.P., whose first term is a and whose common ratio is r.

Then

(1) $S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$.

Multiply both sides by r.

(2) $\therefore \; rS_n = ar + ar^2 + \ldots \qquad + ar^{n-1} + ar^n$.

Subtracting the second series from the first,

$$S_n - rS_n = a - ar^n$$

$$\therefore \quad S_n(1 - r) = a - ar^n$$

$$\therefore \quad S_n = \frac{a - ar^n}{1 - r}$$

$$\therefore \quad \mathbf{S_n = \frac{a(1 - r^n)}{1 - r}} \qquad \ldots \ldots \quad \text{(A)}$$

or by subtracting the first series from the second,

$$\mathbf{S_n = \frac{a(r^n - 1)}{r - 1}} \qquad \ldots \ldots \quad \text{(B)}$$

Note.—It is better to use (A) when r is a proper fraction or negative.

9. Infinite Geometric Series

(1) The Approach to Infinity

When the common ratio of a Geometric Series is numerically greater than unity, as in

$$1, 2, 4, 8, \ldots$$
$$2{\cdot}5, 7{\cdot}5, 22{\cdot}5, \ldots$$

the terms increase in magnitude. The sum of n terms will, therefore, also increase as n increases.

If the number of terms increases without limit, *i.e.*, n is greater than any number we may select, however great, then the sum of these terms will also increase without limit.

This we may express by saying that as n, the number of terms, approaches infinity, S_n, the sum of these terms also approaches infinity. This may also be expressed in the following notation.

If $\qquad n \longrightarrow \infty$, then $S_n \longrightarrow \infty$.

(2) A Decreasing Series

If, however, the common ratio is numerically less than unity, as in such series as

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8} \ldots$$
$$0{\cdot}3, 0{\cdot}03, 0{\cdot}003, 0{\cdot}0003 \ldots$$

then as the number of terms increases, the terms themselves decrease. Using the terms employed above, we may say that as n increases without limit, the terms themselves decrease without limit, and finally become indefinitely small.

We cannot say, however, that the sum of these terms decreases without limit, as n increases without limit. This is a matter for further investigation.

(3) Recurring Decimals

Let us consider the case of what is termed a recurring decimal. We know from arithmetic that

$\frac{1}{3} = 0{\cdot}3333\ .\ .\ .$ to any number of places.

i.e., $\frac{1}{3} = 0{\cdot}3 + 0{\cdot}03 + 0{\cdot}003 + 0{\cdot}0003 + \ .\ .\ .$ to any number of terms.

The series on the right-hand side is seen to be a Geometric Series, with common ratio $\frac{1}{10}$.

There is no limit to the number of terms of this series; at the same time, it is evident that the sum of this series is $\frac{1}{3}$.

Let us now find the sum of a finite number of terms of the series.

For example,
$$S_2 = \frac{33}{100}$$
$$S_3 = \frac{333}{1000}$$
$$S_4 = \frac{3333}{10000}$$
$$.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$

From these and other similar cases, it is clear that the difference between $\frac{1}{3}$ and the various sums S_2, S_3, S_4, . . . is diminishing as the number of terms is increased. We come to the conclusion that the greater the number of terms we take, the more nearly does S_n approach to equality with $\frac{1}{3}$, and that it can never be greater than $\frac{1}{3}$.

Using our previous notation, we can express the result thus

as $$n \longrightarrow \infty,\ S_n \longrightarrow \tfrac{1}{3}.$$

There is thus a *limit* to which S_n approaches and which it cannot exceed.

Clearly all series representing recurring decimals will lead to similar results.

(4) A Graphical Illustration

Another special case which serves to illustrate this approach of the sum of a series to a limit may be seen if we represent graphically the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots$$

Let the rectangle ABCD (Fig. 49) represent a unit area.

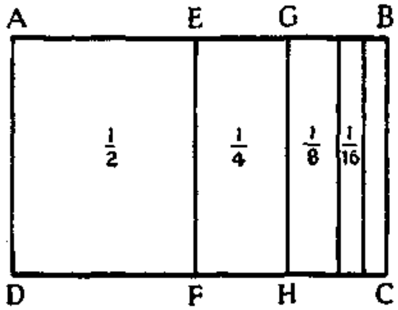

FIG. 49.

If EF be drawn as shown, bisecting the rectangle, then AEFD represents $\tfrac{1}{2}$.

Similarly bisecting EFCB, we get EFHG representing $\tfrac{1}{4}$. Continuing this process of bisecting the rectangle left over in each case, we get a series of rectangles which represent the terms of the series above, viz.:

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots$$

Clearly these rectangles are diminishing as we represent more and more terms of the series in this way.

It is also clear that the sum of all these rectangles, as more and more are taken, is approaching the area of the whole rectangle, *i.e.*, 1, and can never exceed this. Consequently 1 is a limit which the sum of the series approaches as the number of terms is increased without limit, but which it can never exceed.

If the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \ldots$$

be summed by using the formula:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

we get:

$$S_n = \frac{\frac{1}{2}\{1 - (\frac{1}{2})^n\}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{2}\{1 - (\frac{1}{2})^n\}}{\frac{1}{2}}$$

i.e.,
$$S_n = 1 - (\tfrac{1}{2})^n.$$

Examining this result, we see that the term $(\frac{1}{2})^n$ is decreasing as n increases.

If n be increased without limit, then $(\frac{1}{2})^n$ decreases without limit—*i.e.,* it vanishes.

Then we have the result that $S_n \longrightarrow 1$, as $n \longrightarrow \infty$.

(5) The Sum to Infinity

We must now proceed to a general treatment of the problem.

Using the formula previously quoted, viz.:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

i.e.,
$$S_n = \frac{a - ar^n}{1 - r}$$

we have:
$$S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

or
$$S_n = \frac{a}{1 - r} - a \cdot \frac{r^n}{1 - r}$$

Considering the second term of the right-hand side, if r be a proper fraction (*i.e.,* it lies between $+1$ and -1)

then r^n diminishes as n increases, or with the previous notation,

as $$n \longrightarrow \infty,\ r^n \longrightarrow 0$$

Consequently, as $n \longrightarrow \infty$, $a \cdot \dfrac{r^n}{1-r} \longrightarrow 0$.

Thus the right-hand side approaches $\dfrac{a}{1-r}$ as a limit.

This is called the "*sum to infinity*" of the series.

If it be represented by S_∞,

then $$\mathbf{S_\infty = \frac{a}{1-r}}.$$

Examples

(1) *Sum to infinity* $2 + \frac{1}{2} + \frac{1}{8} \ldots$

Here $a = 2$ and $r = \frac{1}{4}$

$$\therefore \quad S_\infty = \frac{2}{1-\frac{1}{4}}$$
$$= \frac{2}{\frac{3}{4}}$$
$$= 2\tfrac{2}{3}.$$

(2) *Find the sum to infinity of the series*, $5, -1, \frac{1}{5}, \ldots$

Here $a = 5,\ r = -\frac{1}{5}$

$$\therefore \quad S_\infty = \frac{5}{1-(-\frac{1}{5})}$$
$$= \frac{5}{1\frac{1}{5}}$$
$$= 4\tfrac{1}{6}.$$

(3) *£500 is invested at 5% per annum, compound interest. What will be the amount of principal and interest together at the end of 10 yrs.?*

In 1 yr. each £1 gains £0·05 interest, *i.e.*, in 1 yr. £1 amounts to £1·05.

$\therefore$ in 1 yr. £500 will amount to £500(1·05).

$\therefore$ £500(1·05) is the amount at the end of first year.

During the second year, each £1 amounts to £1·05.

$\therefore$ During the second year £500(1·05) will amount to £$500(1·05)^2$.

Similarly amount at end of third year is £$500(1·05)^3$.
and amount at end of tenth year is £$500(1·05)^{10}$

or
$$\begin{aligned} A_{10} &= 500(1·05)^{10} \\ &= £814·7 \\ &= \textit{£814 14s.} \end{aligned}$$

(4) *A man pays an Insurance Premium of £20 at the beginning of each year. At the end of 25 yrs. he is to receive all his premiums, together with 3% per annum compound interest. What should he receive, to the nearest £1?*

The first £20 earns interest for 25 yrs.

$\therefore$ It becomes worth £$20(1·03)^{25}$ (see previous example).

The second £20 earns interest for 24 yrs.

$\therefore$ It amounts to £$20(1·03)^{24}$.

Similarly, the third £20 amounts to £$20(1·03)^{23}$, etc., etc.

The last £20 paid earns interest for 1 yr.

$\therefore$ It amounts to £20(1·03).

$\therefore$ Total amount

$$\begin{aligned} &= £20\{1·03 + 1·03^2 + \ldots 1·03^{25}\} \\ &= £20 \times \text{sum of twenty-five terms of above series} \\ &= £20 \times \frac{1·03(1·03^{25} - 1)}{1·03 - 1} \\ &= \textit{£748 to nearest £} \end{aligned}$$

(within limits of four-fig. logs.).

(5) *A marble dropped on a stone floor bounces up a distance equal to 0·85 of the height from which it fell. If it was dropped from a height of 8 ft., how far would it have travelled altogether in its up-and-down movements, when it reached the floor for the tenth time?*

During its first fall it drops 8 ft.

Then it bounces up $(0·85 \times 8)$ ft. and falls down $(0·85 \times 8)$ ft.

$\therefore$ Between its first and second contact with the floor it has travelled $2 \times 0·85 \times 8$ ft.

After its second contact with floor it bounces up $0{\cdot}85 \times 0{\cdot}85 \times 8$ ft., and then falls down $0{\cdot}85 \times 0{\cdot}85 \times 8$ ft.

$\therefore$ Between its second and third contact with the floor it has travelled $2 \times 0{\cdot}85^2 \times 8$ ft., and so on.

Between its ninth and tenth contact with the floor it has travelled $2 \times 0{\cdot}85^9 \times 8$ ft.

$\therefore$ Total distance travelled

$$= 8 + (2 \times 0{\cdot}85 \times 8) + (2 \times 0{\cdot}85^2 \times 8) + \ldots (2 \times 0{\cdot}85^9 \times 8) \text{ ft.}$$

$$= 8 + (2 \times 8)\{0{\cdot}85 + 0{\cdot}85^2 + \ldots (0{\cdot}85)^9\}$$

$$= 8 + 16\{4{\cdot}355\}$$

$$= 8 + 69{\cdot}680$$

$$= \textit{77·68 ft.}$$

EXERCISE 12

1. Find the fifth term of 1, 1·3, 1·69, . . .
2. Find the tenth term of 50, − 25, 12½, . . .
3. Find the ninth term of 1·1, 1·21, 1·331, . . .
4. Find the eighth term of − 0·5, + 0·15, − 0·045, . . .
5. Find the sixth term of 2·2, − 0·55, 0·1375, . . .
6. Find the fifth term of a G.P. whose fourth term is 5, and whose seventh term is 320.
7. The third term of a G.P. is 4·5 and the ninth is 16·2. Find the sixth term.
8. Find the seventh term of a Geometric Series, if the sixth term is 1·09, and the tenth 0·12.
9. Insert two Geometric Means between 100 and 51·2.
10. Insert three Geometric Means between 75 and 30·72.
11. Find the sum of the first six terms of 1, 1·4, 1·96, . . .
12. Find the sum of the first eight terms of 30, − 15, 7½, . . .
13. Find the sum of the first seven terms of 14, − 7, 3½, . . .
14. Find the sum of the first twelve terms of 4, 5, 6¼, . . .

15. Evaluate $6 - 4 + 2\frac{2}{3} \ldots$ to 6 terms.

16. Sum $-12 + 9{\cdot}6 - 7{\cdot}68 \ldots$ to six terms.

17. Sum to infinity $30 + 10 + 3\frac{1}{3} \ldots$

18. Sum to infinity $16 - 8 + 4 \ldots$

19. Evaluate $4 + 1 + \frac{1}{4} \ldots$ *ad. inf.*

20. Evaluate $0{\cdot}\dot{5}$, $0{\cdot}\dot{3}\dot{6}$ and $3{\cdot}8\dot{3}$.

21. A certain ball when dropped to the ground rebounds to $\frac{4}{5}$ of the height from which it falls. If it is dropped from a height of 10 ft., find the total distance it travels before finally coming to rest.

22. At the end of a certain year a tree was 30 ft. high. During the next year it grew 4 ft., and in each succeeding year its rate of growth was $\frac{9}{10}$ of what it was in the previous year. Find its greatest height.

23. A man receives a royalty of £200 with the condition that in each succeeding year it shall be $\frac{19}{20}$ of what it was in the previous year. What will he have received in 8 yrs. ?

24. The yearly output of a silver mine is found to be decreasing by 25% of its previous year's output. If in a certain year its output was £25,000, what could be reckoned as its total future output ?

25. A heavy ball is suspended at the end of a string 6 ft. long. It is drawn to one side so that the string makes an angle of 30° with the vertical. When released, it swings to and fro, each swing in one direction being 1% less than the previous one in the opposite direction. What distance will the ball have travelled before it finally comes to rest ?

CHAPTER 8

THE TRIGONOMETRICAL RATIOS

1. The trigonometrical ratios of an acute angle having been dealt with in Book I, it will be the purpose of this chapter to extend their application to angles greater than 90°.

The Sine

2. Let a line OP (Fig. 50) rotating round O in an anti-clockwise direction mark out a complete rotation from

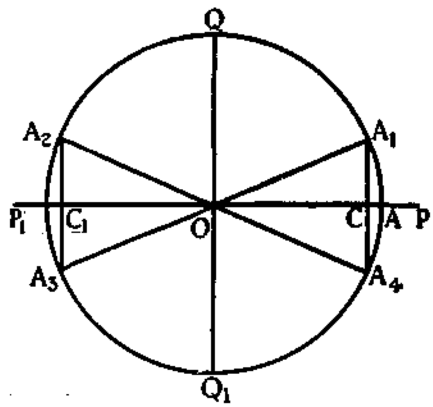

Fig. 50.

0° to 360° so that any point such as A on the line will move on the circumference of a circle.

Then, by drawing QOQ_1, perpendicular to PP_1, the circle is divided into four quadrants.

Let A_1, A_2, A_3 and A_4 be positions of the point A in the four successive quadrants.

Then A_1OP, A_2OP, A_3OP (reflex) and A_4OP (reflex) are angles in the first, second, third and fourth quadrants respectively.

Note.—For convenience, later, the angles A_1OP, A_2OP_1, A_3OP_1 and A_4OP are made equal to one another, but this is not essential for definitions.

Let A_1OP be denoted by θ.

In each quadrant draw perpendiculars to PP_1. Then in each quadrant, the ratio of this perpendicular to the rotating radius OA, etc., is the sine of the corresponding angle.

Thus

in the first quadrant, $\dfrac{A_1C}{OA_1} = \sin A_1OP$

in the second quadrant, $\dfrac{A_2C_1}{OA_2} = \sin A_2OP$

in the third quadrant, $\dfrac{A_3C_1}{OA_3} = \sin A_3OP$ (reflex)

in the fourth quadrant $\dfrac{A_4C}{OA_4} = \sin A_4OP$ (reflex).

In accordance with the usual convention, A_1C and A_2C_1 are taken as positive, being perpendiculars to and standing above the line PP_1, while A_3C_1 and A_4C are negative, being perpendiculars to PP_1 but below it. The rotating line OA is taken as positive in all positions.

3. The Sine in the First Quadrant

Since the sine in this quadrant is given by $\dfrac{A_1C}{OA_1}$, we note:

(1) It is always positive in this quadrant.

(2) Since A_1C increases as the angle increases and OA_1 is constant,

$\therefore$ the sine increases as the angle increases.

(3) When the rotating line reaches OQ, the perpendicular A_1C is equal to OA_1 and therefore,

$$\sin 90^\circ = 1.$$

4. The Sine in the Second Quadrant

For each position of the rotating line, the sine is given by $\frac{A_2C_1}{OA_2}$.

A_2C_1 *decreases* as the angle increases, while OA_2 is constant.

Also A_2C_1 and OA_2 are always both positive.

We therefore conclude that:

(1) The sine is always positive in the second quadrant.

(2) The sine *decreases* as the angle increases.

(3) As OA_2 reaches OP_1, A_2C_1 disappears, then

$$\sin 180° = 0.$$

(4) If A_2OP_1 be taken equal to A_1OP (*i.e.*, θ) then the Δs A_2OC_1 and A_1OC are equal in all respects and $A_2C_1 = A_1C$

$$\therefore \sin A_2OP = \frac{A_2C_1}{OA_2} = \frac{A_1C}{OA_1} = \sin A_1OP$$

$$\therefore \sin (180° - \theta) = \sin \theta$$

i.e., *the sine of an angle is equal to the sine of its supplement.*

5. The Sine in the Third Quadrant

In this quadrant the sine is $\frac{A_3C_1}{OA_3}$.

Here A_3C_1 increases in length as the angle increases, while OA_3 is constant; but as previously stated, A_3C_1 is negative, while OA_3 is positive.

$\therefore$ We conclude that

(1) The sine is negative in this quadrant.

(2) The sine becomes greater numerically as the angle increases.

(3) When OA_3 reaches the position OQ_1, the perpendicular A_3C_1 becomes equal to OA_3 and thus

$$\sin 270^\circ = -1.$$

(4) If A_3OP_1 be taken equal to A_1OP (*i.e.*, $= \theta$) then the Δs A_3OC_1 and A_1OC are equal in all respects, and A_3C_1 is equal in length to A_1C (but A_3C_1 is negative).

$$\therefore \sin A_3OP \text{ (reflex)} = \frac{A_3C_1}{OA_3} = \frac{-A_1C}{OA_1} = -\sin A_1OP$$

$$\therefore \sin (180^\circ + \theta) = -\sin \theta.$$

6. The Sine in the Fourth Quadrant

The sine is now measured by $\frac{A_4C}{OA_4}$, where A_4C is negative, while OA_4 is positive and constant. As the angle increases (*i.e.*, as OA_4 approaches nearer to OP), A_4C decreases in length.

Thus

(1) The sine in this quadrant is negative.

(2) It becomes less numerically, and thus increases as the angle increases.

(3) When OA_4 reaches the position OP, the perpendicular A_4C vanishes

$$\therefore \sin 360^\circ = 0.$$

(4) If A_4OP (acute) be taken equal to A_1OP (*i.e.*, $= \theta$), then the Δs A_4OC and A_1OC are equal in all respects, and A_4C is equal in length (but opposite in sign) to A_1C.

$$\therefore \sin A_4OP \text{ (reflex)} = \frac{A_4C}{OA_4} = \frac{-A_1C}{OA_1} = -\sin A_1OP$$

$$\therefore \sin (360^\circ - \theta) = -\sin \theta.$$

We thus see that the sine is positive in the first and second quadrants only.

7. Negative Rotation and Negative Angles

It should also be noted that OP may reach the position OA_4, OA_3, etc., by a rotation in a clockwise direction.

Such a rotation may be regarded as yielding a negative angle; but however OA_4 is reached—whether by an anti-clockwise (positive) rotation of magnitude A_4OP, or any number of complete revolutions together with A_4OP, or by a clockwise (negative) rotation of magnitude A_4OP, or any number of complete (negative) revolutions together with A_4OP—the sine of the angle thus formed will be the same as that of sin A_4OP reflex itself. This will also apply to cosine and tangent values.

Thus we see that

$$\sin\theta = -\sin(-\theta).$$

Examples

(1) *Write down the values of sin* 55°, *sin* 128°, *sin* 219°, *sin* 324°, *sin* (— 100°).

(*a*) $\sin 55° = 0{\cdot}8192$ (direct from Sine Tables).

(*b*) $\sin 128° = \sin(180 - 52)° = \sin 52° = 0{\cdot}7880.$

(*c*) $\sin 219° = \sin(180 + 39)° = -\sin 39° = -0{\cdot}6293.$

(*d*) $\sin 324° = \sin(360 - 36)° = -\sin 36° = -0{\cdot}5878.$

(*e*) $\sin(-100°) = \sin 260° = \sin(180 + 80)° = -\sin 80°$
$= -0{\cdot}9848.$

(2) *Construct angles whose sines* are $\frac{2}{5}$, $-\frac{3}{5}$, *and* $-\frac{5}{8}$.

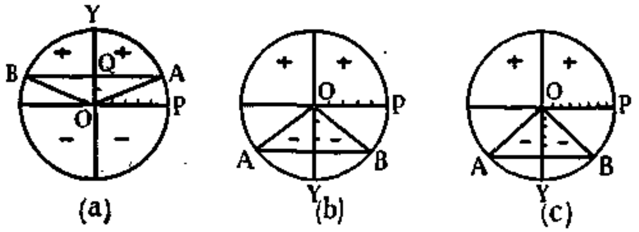

FIG. 51.

(*a*) *Angles whose Sines are* $\frac{2}{5}$ [Fig. 51(*a*)].

Draw a circle of suitable size with centre O and divide into four quadrants in the usual way. Divide radius OP into five equal parts, and on the perpendicular OY measure up from O a distance OQ equal to two of these equal parts. Through Q draw BA parallel to OP.

Join OA, OB.

Then sin AOP $= \frac{3}{5}$ and sin BOP $= \frac{3}{5}$.

In the other quadrants, the sine would be negative.

$\therefore$ AOP and BOP are the required angles.

(b) *Angles whose Sines are* $-\frac{3}{5}$ [Fig. 51(b)].

Using a similar construction as before, we must now place the angles in the third and fourth quadrants, so as to obtain a negative value for the sine. The construction shows that AOP and BOP (both reflex) are the required angles.

(c) *Angles whose Sines are* $-\frac{5}{8}$ [Fig. 51(c)].

The figure shows eight equal parts in OP and five of equal size on OY. The required angles are AOP and BOP (both reflex).

Inverse Notation.

An alternative method of writing "the angle whose sine is $\frac{4}{5}$" is $\sin^{-1}\frac{4}{5}$. Similarly, $\sin^{-1}(-\frac{5}{8})$ would denote "the angle whose sine is $(-\frac{5}{8})$. This notation is known as the "Inverse Notation."

8. Graph of sin θ

We are now in a position to form a mental picture of the variations in $\sin\theta$, by drawing its graph as θ varies from 0° to 360°.

Let $$y = \sin\theta$$

We can use our previous knowledge and the tables to tabulate the following:

$\theta°$	0	20	40	60	80	90	100
$y=\sin\theta$	0	0·3420	0·6428	0·8660	0·9848	1	0·9848

$\theta°$	120	140	160	180	200	220	240
$y=\sin\theta$	0·8660	0·6428	0·3420	0	−0·3420	−0·6428	−0·8660

$\theta°$	260	270	280	300	320	340	360
$y=\sin\theta$	−0·9848	−1	−0·9848	−0·8660	−0·6428	−0·3420	0

On plotting these values, we obtain the graph shown in Fig. 52.

The graph indicates the changes that occur in the value of sin θ as θ increases.

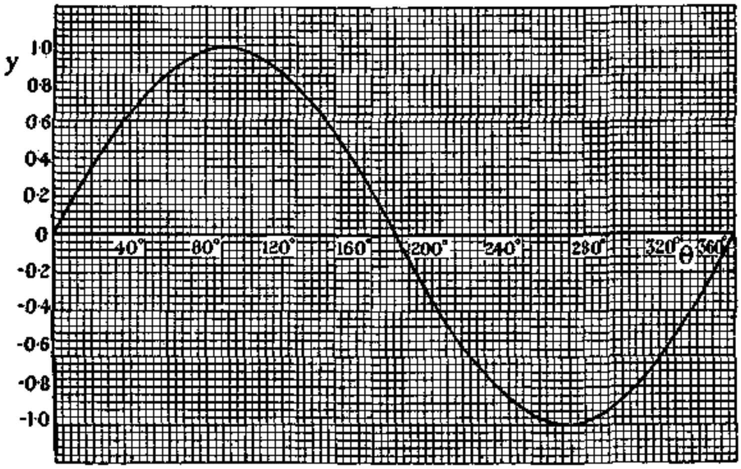

FIG. 52.
Graph of $y = \sin \theta$.

(1) sin θ increases from 0 to 1 as θ increases from 0° to 90°.

(2) sin θ decreases from 1 to 0 as θ increases from 90° to 180°.

(3) sin θ decreases from 0 to − 1 (and is negative) as θ increases from 180° to 270°.

(4) sin θ increases from − 1 to 0 (and is negative) as θ increases from 270° to 360°.

If θ be increased beyond 360°, the curve merely repeats itself in loops exactly similar to those shown.

Example

Use the sine curve to solve the equation

$$\sin x = 0{\cdot}7$$

Through 0·7 on the y scale, draw a horizontal line cutting the curve in two points. Dropping perpendiculars to the x axis, we read the values of x, which are 44° and 136° (to nearest degree).

THE COSINE

9. Using Fig. 50 as previously described, we now proceed to examine the cosine in the four different quadrants.

By definition,

$$\begin{aligned}
\cos A_1OP \quad &= \frac{OC}{OA_1} \text{ in the first quadrant}\\
\cos A_2OP \quad &= \frac{OC_1}{OA_2} \text{ in the second quadrant}\\
\cos A_3OP \text{ (reflex)} &= \frac{OC_1}{OA_3} \text{ in the third quadrant}\\
\cos A_4OP \text{ (reflex)} &= \frac{OC}{OA_4} \text{ in the fourth quadrant}
\end{aligned}$$

In accordance with the usual convention,
OC is positive (since it is measured to the right of QQ_1)
OC_1 is negative (since it is measured to the left of QQ_1).

10. The Cosine in the First Quadrant

Since $\cos A_1OP = \dfrac{OC}{OA_1}$, we note that

(1) It is always positive, since OC and OA_1 are both positive.

(2) Since OC decreases as the angle A_1OP increases, the cosine decreases as the angle increases.

(3) When the rotating line reaches the position OQ, OC vanishes.

$$\therefore \cos 90° = 0.$$

11. The Cosine in the Second Quadrant

For each position of the rotating line, such as OA_2, the cosine is given by $\frac{OC_1}{OA_2}$.

Since OC_1 which is negative, increases in length as the angle increases, OA_2 being constant and positive, it follows that

(1) The cosine is always negative in the second quadrant.

(2) The cosine increases numerically as the angle increases.

(3) When OA_2 reaches the position OP_1, $OC_1 = OA_2$

$$\therefore \textit{cos } 180^\circ = -1.$$

(4) If A_2OP_1 be taken equal to A_1OP (*i.e.*, θ), then Δs A_2OC_1 and A_1OC are equal in all respects, and

$$OC_1 = -OC$$

$$\therefore \cos A_2OP = \frac{OC_1}{OA_2} = \frac{-OC}{OA_1} = -\cos A_1OP$$

$$\therefore \cos(180^\circ - \theta) = -\cos\theta$$

or *the cosine of an angle = — cosine of its supplement.*

12. The Cosine in the Third Quadrant

Here the cosine is given by $\frac{OC_1}{OA_3}$.

As the angle increases, OC_1 diminishes, while OA_3 is constant, OC_1 still being negative and OA_3 positive.

Thus

(1) The cosine is always negative in this quadrant.

(2) Its numerical value diminishes as the angle increases.

(3) When OA_3 reaches the position OQ_1, OC_1 vanishes

$$\therefore \cos 270^\circ = 0.$$

(4) If A_3OP_1 be taken equal to the angle A_1OP (*i.e.*, θ), then Δs A_3OC_1 and A_1OC are equal in all respects, and $OC_1 = -OC$

$$\therefore \cos A_3OP = \frac{OC_1}{OA_3} = \frac{-OC}{OA_1} = -\cos A_1OP$$

$$\therefore \cos (180^\circ + \theta) = -\cos \theta.$$

13. The Cosine in the Fourth Quadrant

Here the cosine is given by $\frac{OC}{OA_4}$, where both OC and OA_4 are positive.

As the angle A_4OP (reflex) increases, OC increases in length.

Thus

(1) The cosine is always positive in this quadrant.

(2) Its value increases as the angle increases.

(3) When OA_4 reaches the position OP, $OC = OA_4$

$$\therefore \cos 360^\circ = 1.$$

(4) If A_4OP be taken equal to A_1OP (*i.e.*, θ), then the Δs A_4OC and A_1OC are equal in all respects, and OC is common to both Δs

$$\therefore \cos A_4OP \text{ (reflex)} = \frac{OC}{OA_4} = \frac{OC}{OA_1} = \cos A_1OP$$

$$\therefore \cos (360^\circ - \theta) = \cos \theta$$

Also $\qquad \cos \theta = \cos (-\theta).$

We note that the cosine is positive in the first and fourth quadrants only.

Examples

Write down the following cosine values.

(1) $\cos 72^\circ 15' = 0{\cdot}3049$.

Note.—Since, as the angle increases in the first quadrant, the cosine decreases, the " mean differences " given in the Cosine Tables must be subtracted, instead of being added as in the case of other tables. This applies throughout the four quadrants.

(2) $\cos 172^\circ 15' = \cos (180^\circ - 7^\circ 45') = -\cos 7^\circ 45' = -0{\cdot}9909$.

(3) $\cos 241° 22' = \cos (180° + 61° 22') = - \cos 61° 22' = - 0{\cdot}4792.$

(4) $\cos 331° = \cos (360° - 29°) = + \cos 29° = 0{\cdot}8746.$

(5) $\cos (- 110°) = \cos (360° - 110°) = \cos 250°$
$= \cos (180° + 70°) = - \cos 70° = - 0{\cdot}3420.$

(6) *Draw an obtuse angle whose cosine is $- \frac{3}{5}$, and find its size by the use of tables.*

Draw a circle of radius five units. On the horizontal diameter set off OC equal to three of the units used.

At C erect the perpendicular CA. Then AOP is the required angle (Fig. 53).

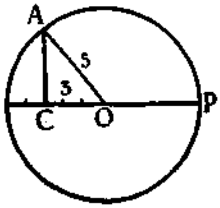

FIG. 53.

To find its actual size in degrees, find, in the Cosine Table, the angle whose cosine is $\frac{3}{5}$ or 0·6. This gives 53° 8′ (to nearest min.). But in the second quadrant

$$\cos (180° - \theta) = - \cos \theta$$

$$\therefore \cos (180° - 53° 8') = - \cos 53° 8' = - 0{\cdot}6$$

$\therefore$ The required angle is $180° - 53° 8' = 126° 52'$.

14. Graph of cos θ

To plot the graph of $y = \cos \theta$, we tabulate in the usual way:

θ	0	20	40	60	80	90	100
$y = \cos \theta$	1	0·9397	0·7660	0·5000	0·1736	0	−0·1736

θ	120	140	160	180	200	220	240
$y = \cos \theta$	−0·5000	−0·7660	−0·9397	−1	−0·9397	−0·7660	−0·5000

θ	260	270	280	300	320	340	360
$y = \cos \theta$	−0·1736	0	0·1736	0·5000	0·7660	0·9397	1

Fig. 54 shows the graph obtained on plotting these values. It will be seen to be of a similar shape to the graph of $y = \sin\theta$, but starting with a value of 1 when $x = 0$, whereas $y = \sin\theta$ has a value of 0 when $x = 0$. Not only is the actual graph visualised, but it can be used to read off the cosines of other angles in addition to those tabulated, and also to find angles whose cosines are given.

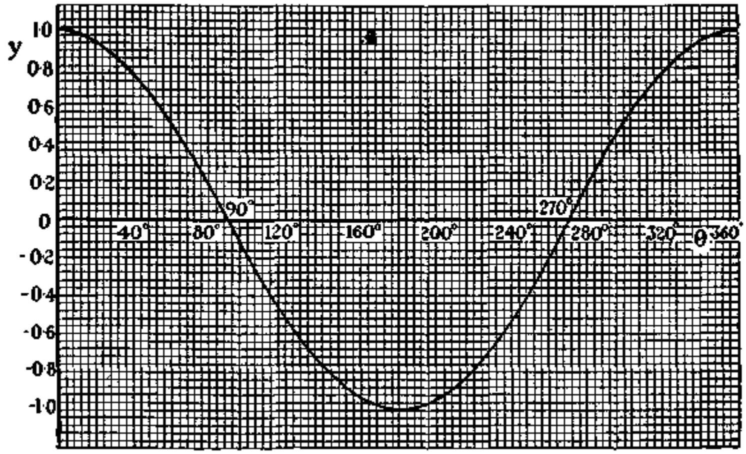

FIG. 54. Graph of $y = \cos\theta$.

Example

On the graph of $y = \cos\theta$, read off the values of the angles whose cosine is $-0{\cdot}4$.

By inspection we see that the cosine is $-0{\cdot}4$ in two places —namely, when $\theta = 114°$ and $246°$ (to the nearest degree in each case).

THE TANGENT

15. Let OP rotate as before into positions such as OA_1, OA_2, etc. Draw the perpendiculars N_1N_4, N_2N_3 as in the figure (Fig. 55).

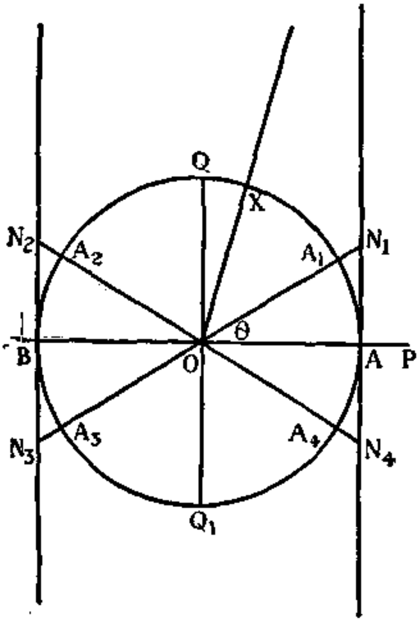

FIG. 55.

Let the angle formed as the line OP rotates be represented by θ.

$$\text{In the first quadrant, } \tan A_1OP = \frac{N_1A}{OA}$$

$$\text{In the second quadrant, } \tan A_2OP = \frac{N_2B}{OB}$$

$$\text{In the third quadrant, } \tan A_3OP \text{ (reflex)} = \frac{N_3B}{OB}$$

$$\text{In the fourth quadrant, } \tan A_4OP \text{ (reflex)} = \frac{N_4A}{OA}$$

The conventions previously used for positive and negative distances will be used here.

16. The Tangent in the First Quadrant

In this quadrant $\tan\theta = \frac{N_1A}{OA}$, where N_1A and OA are both positive.

As θ increases, N_1A increases, while OA remains constant.

Hence we see that

(1) The tangents of angles in the first quadrant are positive.

(2) As the angle θ increases, $\tan\theta$ increases.

It is clear that as θ approaches near to 90°, N_1A becomes very large, and as OA remains constant, the value of $\frac{N_1A}{OA}$ —*i.e.*, $\tan\theta$—becomes very large. As θ approaches infinitely near to 90°, the point N_1 is at an infinitely great distance and $\tan\theta$ becomes indefinitely large. This is expressed by saying that as θ approaches to coincidence with 90°, $\tan\theta$ becomes infinitely large.

Or, using the symbols on p. 127, we may say that

$$\text{as } \theta \longrightarrow 90°, \tan\theta \longrightarrow \infty$$

This is what is meant when we say

$$\tan 90° = \infty.$$

17. The Tangent in the Second Quadrant

In this quadrant, the tangent may be measured by $\frac{N_2B}{OB}$ where N_2B is positive, while OB is negative : the tangent will therefore be negative.

As the angle increases, N_2B decreases, while OB remains constant.

When the angle is greater than 90° by an exceedingly small amount, as in the case of an angle a very small amount less than 90°, N_2B will be exceedingly large, and consequently the tangent will be exceedingly large numerically.

but will be negative as OB is negative. Hence, being negative, the absolute value of $\tan\theta$ is increasing through negative values as θ increases, until when the rotating line coincides with OB, N_2B vanishes and $\tan\theta = 0$.

Thus in the second quadrant we may say that

(1) The tangent is always negative.

(2) The tangent increases from $-\infty$ to 0.

(3) *tan 180° = 0.*

(4) If A_2OB be taken equal to $A_1OP[=\theta]$, then the Δs N_2OB and N_1OA are equal in all respects.

$$\therefore\ N_2B = N_1A$$

and $\quad OB = -OA$

$$\therefore\ \tan A_2OP = \frac{N_2B}{OB} = \frac{N_1A}{-OA} = -\tan A_1OP$$

$$\therefore\ \tan(180^\circ - \theta) = -\tan\theta$$

or *the tangent of an angle = — tangent of its supplement.*

18. The Tangent in the Third Quadrant

Here the tangent is given by $\frac{N_3B}{OB}$ where both N_3B and OB are negative.

As the angle increases, N_3B increases, and when the revolving line reaches OQ_1, N_3B becomes infinitely large (but negative).

We thus see that:

(1) The tangent in the third quadrant is always positive.

(2) As the angle increases, the tangent increases.

(3) *tan 270° = ∞.*

(4) If A_3OB be taken equal to $A_1OP\ (=\theta)$, the Δs N_3OB and N_1OA are equal in all respects.

$\therefore\ N_3B = -N_1A$ (since N_3B is negative, and N_1A is positive) and $OB = -OA$ (OB being negative, while OA is positive).

$$\therefore \tan A_3OP \text{ (reflex)} = \frac{N_3B}{OB} = \frac{-N_1A}{-OA} = \frac{N_1A}{OA} = \tan A_1OP$$

$$\therefore \textit{tan } (180^\circ + \theta) = \textit{tan } \theta.$$

19. The Tangent in the Fourth Quadrant

Here the tangent is measured by $\frac{N_4A}{OA}$, where N_4A is negative, while OA is positive.

As the angle increases, N_4A decreases, while OA is constant. When OA_4 coincides with OA, N_4A vanishes.

We thus see that :

(1) The tangent in the fourth quadrant is always negative.

(2) The larger the angle, the smaller the numerical value of the tangent; and as it is negative, it is increasing as the angle increases.

(3) *tan* $360^\circ = 0$.

(4) If A_4OP be taken equal to A_1OP $[= \theta]$, then the Δs N_4OA and N_1OA are equal in all respects.

$\therefore$ $N_4A = -N_1A$ while OA is common to both Δs.

$$\therefore \quad \tan A_4OP \text{ (reflex)} = \frac{N_4A}{OA} = \frac{-N_1A}{OA} = -\tan A_1OP$$

$$\therefore \textit{tan } (360^\circ - \theta) = -\textit{tan } \theta$$

$$\textit{Also tan } \theta = -\textit{tan } (-\theta).$$

We note that the tangent is positive only in the first and third quadrants.

Examples

(1) tan $63^\circ = 1{\cdot}9626$ (direct from tables).
(2) tan $125^\circ = \tan (180^\circ - 55^\circ) = -\tan 55^\circ = -1{\cdot}4281$.
(3) tan $226^\circ = \tan (180^\circ + 46^\circ) = \tan 46^\circ = +1{\cdot}0355$.
(4) tan $342^\circ = \tan (360^\circ - 18^\circ) = -\tan 18^\circ = -0{\cdot}3249$.
(5) tan $(-76^\circ) = -\tan 76^\circ = -4{\cdot}0108$.

20. Graph of tan θ

The following values are tabulated and plotted in the usual way:

θ	0	30	60	90	120	150	180
$y = \tan\theta$	0	0·5774	1·7321	∞	−1·7321	−0·5774	0

θ	210	240	270	300	330	360
$y = \tan\theta$	0·5774	1·7321	∞	−1·7321	−0·5774	0

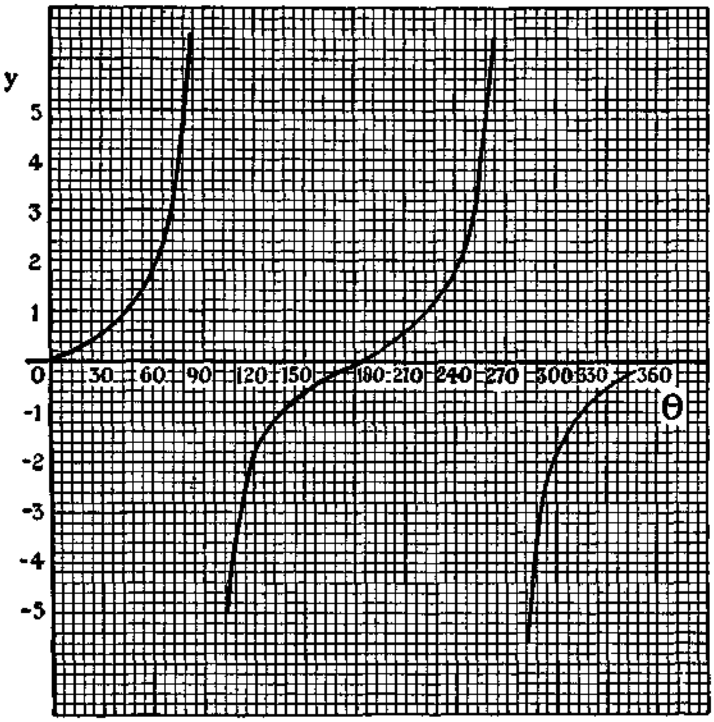

FIG. 56. Graph of $y = \tan\theta$.

The graph (Fig. 56) shows (*a*) the increase of the tangent value as θ increases from $0°$ to $90°$, (*b*) the instantaneous change from $+\infty$ to $-\infty$ at $90°$ and $270°$, (*c*) after passing $180°$, the tangent curve repeats itself (this would go on indefinitely beyond $360°$).

It will have been noted that in finding the sine, etc., of an angle greater than $90°$, the method involved finding the angle between the rotating line and the line P′P. The sine, etc., of this angle could then be read from the appropriate table, which would give the numerical value of the required function. It would only remain to add the correct algebraical sign (+ or −).

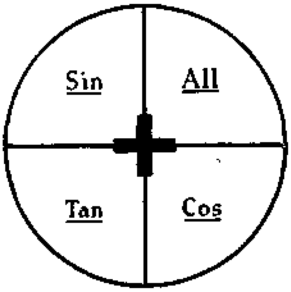

FIG. 57. Functions which are positive in each Quadrant.

By examination of the graphs, it will be found that only those ratios shown in Fig. 57 are positive in the respective quadrants (as indicated by the + sign at the centre). All the others are negative.

21. Other Trigonometrical Ratios

In addition to the sine, cosine, and tangent of an angle, there are three other commonly used ratios, which are merely the reciprocals of the former :

$\dfrac{1}{\text{sine}\,\theta}$ is called the *cosecant* of the angle θ, and is written *cosec* θ

$\dfrac{1}{\text{cosine}\,\theta}$ is called the *secant* of the angle θ, and is written *sec* θ

$\dfrac{1}{\text{tangent}\,\theta}$ is called the *cotangent* of the angle θ, and is written *cot* θ

Thus

$$\sin 60° = 0{\cdot}866 \quad \therefore \quad \operatorname{cosec} 60° = \frac{1}{0{\cdot}866} = 1{\cdot}155$$

$$\cos 35° = 0{\cdot}8192 \quad \therefore \quad \sec 35° = \frac{1}{0{\cdot}8192} = 1{\cdot}221$$

$$\tan 70° = 2{\cdot}7475 \quad \therefore \quad \cot 70° = \frac{1}{2{\cdot}7475} = 0{\cdot}3640$$

In the different quadrants, the cosec, sec and cot must of necessity bear the same algebraic sign as the sin, cos and tan respectively.

22. Other Simple Relations between Ratios

A few other simple connections between the trigonometrical functions are worth considering:

(1) $\tan A = \dfrac{\sin A}{\cos A}$ for all values of A.

Consider the right-angled triangle shown in Fig. 58, with angle A and sides a, b, c.

Here

$$\sin A = \frac{a}{c}$$

$$\cos A = \frac{b}{c}$$

$$\tan A = \frac{a}{b}$$

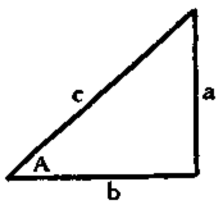

FIG. 58.

$$\therefore \quad \frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b} = \tan A.$$

This is true for all acute angles such as A. In the second, third and fourth quadrants the lengths of a, b and c will be similarly related, and a little consideration will show that the algebraic signs also agree with the above identity.

For example, in the third quadrant,

sin A is negative
cos A is negative

$\therefore \dfrac{\sin A}{\cos A}$ is $\dfrac{\text{negative}}{\text{negative}}$ which gives the necessary positive value for tan A.

(2) *$\sin^2 A + \cos^2 A = 1$* for all values of A.

Using the above figure,

$$\sin^2 A = \frac{a^2}{c^2} \quad \text{and} \quad \cos^2 A = \frac{b^2}{c^2}$$

$$\therefore \quad \sin^2 A + \cos^2 A = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$= \frac{a^2 + b^2}{c^2}$$

But by the Theorem of Pythagoras,

$$a^2 + b^2 = c^2$$

$$\therefore \quad \sin^2 A + \cos^2 A = 1.$$

(3) *$\sec^2 A = 1 + \tan^2 A$.*

In previous figure,

$$\sec^2 A = \frac{1}{\cos^2 A} = \frac{c^2}{b^2} \quad \text{and} \quad \tan^2 A = \frac{a^2}{b^2}$$

$$\therefore \quad 1 + \tan^2 A = 1 + \frac{a^2}{b^2}$$

$$= \frac{b^2 + a^2}{b^2}$$

$$= \frac{c^2}{b^2} \text{ (by Principle of Pythagoras)}$$

$$\therefore \quad 1 + \tan^2 A = \sec^2 A.$$

$\operatorname{cosec}^2 A = 1 + \cot^2 A$

$$\operatorname{cosec}^2 A = \frac{1}{\sin^2 A} = \frac{c^2}{a^2}$$

$$\cot^2 A = \frac{1}{\tan^2 A} = \frac{b^2}{a^2}$$

$$\therefore \quad 1 + \cot^2 A = 1 + \frac{b^2}{a^2}$$

$$= \frac{a^2 + b^2}{a^2}$$

$$= \frac{c^2}{a^2}$$

$$\therefore \quad 1 + \cot^2 A = \operatorname{cosec}^2 A.$$

These relations are true for all values of A, and are so useful that they are worth memorising.

By use of these relations and the method used in obtaining them, other relations of less common occurrence can be proved.

Examples

(1) *Prove that* $\frac{\sec^2 A - 2}{2 - \operatorname{cosec}^2 A} = \tan^2 A$

(*a*) We can use the right-angled triangle of Fig. 58, as before

$$\frac{\sec^2 A - 2}{2 - \operatorname{cosec}^2 A} = \frac{\frac{c^2}{b^2} - 2}{2 - \frac{c^2}{a^2}} = \frac{\frac{c^2 - 2b^2}{b^2}}{\frac{2a^2 - c^2}{a^2}} = \frac{c^2 - 2b^2}{b^2} \times \frac{a^2}{2a^2 - c^2}$$

$\therefore$ Since $\quad a^2 + b^2 = c^2,$

$$\frac{\sec^2 A - 2}{2 - \operatorname{cosec}^2 A} = \frac{a^2 + b^2 - 2b^2}{b^2} \times \frac{a^2}{2a^2 - a^2 - b^2}$$

$$= \frac{a^2 - b^2}{b^2} \times \frac{a^2}{a^2 - b^2}$$

$$= \frac{a^2}{b^2}$$

$$= \tan^2 A.$$

Alternatively, we might have used $\sec^2 = 1 + \tan^2$, etc.

$$\frac{\sec^2 A - 2}{2 - \operatorname{cosec}^2 A} = \frac{1 + \tan^2 A - 2}{2 - (1 + \cot^2 A)}$$
$$= \frac{\tan^2 A - 1}{1 - \cot^2 A}$$
$$= \frac{\tan^2 A - 1}{1 - \dfrac{1}{\tan^2 A}}$$
$$= \frac{\tan^2 A - 1}{\dfrac{\tan^2 A - 1}{\tan^2 A}}$$
$$= \frac{\tan^2 A - 1}{1} \times \frac{\tan^2 A}{\tan^2 A - 1}$$
$$= \textit{tan}^2\, A.$$

(2) *Prove that* $\dfrac{\tan\theta + \tan\phi}{\cot\theta + \cot\phi} = \tan\theta \,.\, \tan\phi.$

$$\frac{\tan\theta + \tan\phi}{\cot\theta + \cot\phi} = \frac{\tan\theta + \tan\phi}{\dfrac{1}{\tan\theta} + \dfrac{1}{\tan\phi}}$$
$$= \frac{\tan\theta + \tan\phi}{\dfrac{\tan\theta + \tan\phi}{\tan\theta \,.\, \tan\phi}}$$
$$= \tan\theta \,.\, \tan\phi.$$

EXERCISE 13

1. Write down the values of the sine, cosine and tangent of:

(a) $102°$.
(b) $242° 32'$.
(c) $315° 20'$.
(d) π radians.
(e) $\dfrac{\pi}{2}$ radians.
(f) $6{\cdot}222$ radians.
(g) $-51°$.
(h) $-300°$.
(i) $-256°$.
(j) $2000°$.
(k) $149° 33'$.
(l) $201° 13'$.
(m) $343° 8'$.

2. Find the value of :

(*a*) cosec 154°. (*c*) cot 321°. (*e*) sec 300°.
(*b*) sec 235°. (*d*) cosec 251°. (*f*) cot 163°.

3. Given that an angle is in the third quadrant, and its sine $= -\frac{2}{5}$, find its cosine and tangent.

4. If $\cos \theta = \dfrac{a^2 - b^2}{2lm}$, find all the possible values of θ up to 360°, when $a = 6$, $b = 7$, $l = 9$, $m = 10$.

5. Given that $2 \sin x = 0{\cdot}96$, find all values of x up to 360°.

6. If $\sin \theta = -\frac{5}{13}$, find all possible values of $\sec \theta$ and $\cot \theta$ without evaluating θ.

7. If $\sec \theta = 5$ (θ being an acute angle) find $\sin \theta$ and $\cot \theta$.

8. If $\sin A = 0{\cdot}5$ (A being between 90° and 180°), find $\sec A + \tan A$ (without evaluating A).

9. If $\sec A + \tan A = 5$ (A being an acute angle), find $\cos A$.

10. The cosine of an acute angle is m; find its cosecant and also its tangent.

11. Evaluate $ae^{-kt} \sin (nt + g)$, when $a = 6$, $t = 0{\cdot}004$, $k = 40$, $n = 1000$, $e = 2{\cdot}718$, $g = 1{\cdot}235$, where $(nt + g)$ is in radians.

12. Evaluate $\sin \dfrac{3\pi}{2} \,.\, \cos \dfrac{\pi}{2} - \cos \dfrac{3\pi}{2} \,.\, \sin \dfrac{\pi}{2}$ (angle is in radians).

13. Evaluate $\sin \dfrac{\pi}{3} \,.\, \cos \dfrac{\pi}{4} + \cos \dfrac{\pi}{3} \,.\, \sin \dfrac{\pi}{4}$ (angle is in radians).

14. Evaluate $\cos \dfrac{\pi}{4} \,.\, \cos \dfrac{\pi}{6} - \sin \dfrac{\pi}{4} \,.\, \sin \dfrac{\pi}{6}$ (angle is in radians).

15. Evaluate $\dfrac{2}{\sec \frac{\pi}{5} \,.\, \operatorname{cosec} \frac{\pi}{5}}$ (angle is in radians).

Prove the following :

16. $\dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos^2 \theta - \sin^2 \theta.$

17. $(1 + \tan^2 A) \cos^2 A = 1.$

18. $\dfrac{2 \tan B}{1 + \tan^2 B} = 2 \sin B \, . \cos B.$

19. $\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x.$

20. $1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1.$

21. $(\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = 1.$

22. $(\sec \theta - \cos \theta) = \sqrt{(\tan \theta + \sin \theta)(\tan \theta - \sin \theta)}.$

23. $\tan \phi + \cot \phi = \sec \phi \, . \operatorname{cosec} \phi.$

24. $\dfrac{1 - \sin A}{1 + \sin A} = \{\sec A - \tan A\}^2.$

25. Graph each of the functions $2{\cdot}5 \sin x^\circ$ and $(2 - \cos x^\circ)$ for values of x from -30° to $+140^\circ$, using the same reference axes for both graphs, and one inch to represent 20°. By means of the graph, solve the equation

$$\cos x^\circ + 2{\cdot}5 \sin x^\circ = 2$$

(N.C.T.E.C., 1934.)

26. On the same axes and to the same scales, graph the functions $3{\cdot}5 \sin x$ and $(2{\cdot}5 - \cos x)$, taking values of x from 0° to 180°. Then using your diagrams, solve the equation

$$3{\cdot}5 \sin x + \cos x = 2{\cdot}5$$

(U.E.I., 1935.)

27. In a calculation on potential, the following formula was employed: $V(0{\cdot}2501 \sin \theta - 0{\cdot}4905 \cdot \cos \theta) = 300$, where $V =$ potential in volts, and $\theta =$ angle of lag. Calculate V for each of the following values of θ, viz., $\theta = 0^\circ$, $\theta = 90^\circ$, $\theta = 158^\circ$. (U.L.C.I., 1927.)

28. The turning effect of a ship's rudder is shown theoretically to be proportional to $(\cos x^\circ - \cos 3x^\circ)$, where x° is the angle the rudder makes with the keel. Plot the function $(\cos x^\circ - \cos 3x^\circ)$ for values of x from 0° to 70°, using for the values of the function as large a scale as pos-

sible, and from the graph estimate the value of x for which the turning effect will be a maximum.

(N.C.T.E.C., 1931.)

29. A vector, OP, rotates round a fixed point O. The length of OP is given by the equation

$$\mathrm{OP} = \tfrac{3}{2} + 1{\cdot}5\{\cos(\theta + 30^\circ)\}$$

Draw the curve traced by the free end of the line OP. Take as values of θ, every 30° from 0° to 360°.

(U.L.C.I., 1928.)

CHAPTER 9

SOLUTION OF TRIANGLES

1. The three sides and three angles of a triangle are sometimes called its "elements"; when we are given a sufficient number of these elements, it is possible to find the remainder; we are then said to "solve the triangle."

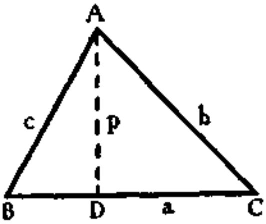

FIG. 59.

We may sometimes draw the triangle to scale according to the given data, and so solve it by actual measurement; but in many cases such a method does not give sufficiently accurate results: much greater accuracy can be obtained by the use of formulæ or "rules," two of which are usually sufficient for solving any triangle.

The notation used in the following paragraphs will be that usually adopted: A, B, C for the angles, *a*, *b*, *c* for the sides opposite these angles respectively (see Fig. 59).

2. The Sine Rule

(a) *Acute-angled Triangle.*

In the Δ ABC, draw AD perpendicular to BC.

Let AD $= p$.

Then $\frac{p}{c} = \sin B \qquad \therefore \quad p = c \sin B$

$\frac{p}{b} = \sin C \qquad \therefore \quad p = b \sin C$

$$\therefore \quad c \sin B = b \sin C$$

or

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

and in a similar way, it could be proved that

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Thus if we know

(1) two sides and one angle

or (2) one side and two angles,

provided that, in each case, an angle and *the side opposite to it* are included, we can use this rule (called the *Sine Rule*) to evaluate the unknown sides and angles: that is, *to solve the triangle.*

(b) *Obtuse-angled Triangle.*

Consider the case of the Δ ABC where C is obtuse.

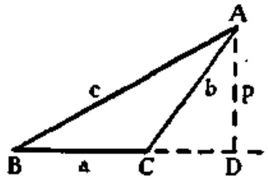

FIG. 60.

Let p be the perpendicular from A to BC produced (Fig. 60).

Then $\quad \frac{p}{c} = \sin B \qquad \therefore \quad p = c \sin B$

and $\quad \frac{p}{b} = \sin ACD \quad \therefore \; p = b \sin ACD$

But $\sin ACD = \sin (180 - ACD) = \sin C.$

$$\therefore \quad p = c \sin B = b \sin C, \text{ as before}$$

or $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Examples (on Sine Rule)

(1) *Solve the triangle* ABC, *given that*

$$A = 70°,\ C = 58°\ 16' \textit{ and } b = 6 \textit{ ins.}$$

The triangle is drawn for reference: it is not drawn to scale (Fig. 62).

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

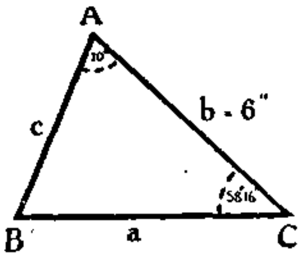

FIG. 62.

As $A = 70°$ and $C = 58°\ 16'$, and b is not opposite either of these angles, it would appear at first sight that the sine rule is inapplicable.

But we know

$$\begin{aligned} B &= 180° - (70° + 58°\ 16') \\ &= 180 - 128°\ 16' \\ &= 51°\ 44' \end{aligned}$$

We can now use

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Substituting the known values,

$$\frac{a}{\sin 70°} = \frac{6 \text{ ins.}}{\sin 51°\ 44'}$$

$$\therefore \quad a = \frac{6 \times \sin 70°}{\sin 51°\ 44'}$$

Taking logs. of both sides,

$$\begin{aligned} \log a &= \log 6 + \log \sin 70° - \log \sin 51°\ 44' \\ &= 0{\cdot}7782 + \bar{1}{\cdot}9730 - \bar{1}{\cdot}8949 \\ &= 0{\cdot}8563 \\ &= \log 7{\cdot}183 \end{aligned}$$

$$\therefore \quad a = 7{\cdot}183 \textit{ ins.}$$

Using $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

and proceeding as before, we obtain,

$$\frac{6}{\sin 51^\circ 44'} = \frac{c}{\sin 58^\circ 16'}$$

$$\therefore \quad c = \frac{6 \times \sin 58^\circ 16'}{\sin 51^\circ 44'}$$

$$\therefore \quad c = 6{\cdot}501 \text{ ins.}$$

Thus the complete solution is

$$c = 6{\cdot}501 \text{ ins}, \ a = 7{\cdot}183 \text{ ins. and } B = 51^\circ 44'.$$

(2) *Solve the triangle* ABC *given that* $c = 85{\cdot}3$ *ft.*, $b = 70{\cdot}25$ *ft. and* $B = 40^\circ$.

Using $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

and substituting the given values,

$$\frac{70{\cdot}25}{\sin 40^\circ} = \frac{85{\cdot}3}{\sin C}$$

$$\therefore \quad \sin C = \frac{85{\cdot}3 \times \sin 40^\circ}{70{\cdot}25}$$

Taking logs of both sides

$$\begin{aligned} \log \sin C &= \log 85{\cdot}3 + \log \sin 40^\circ - \log 70{\cdot}25 \\ &= 1{\cdot}9309 + \bar{1}{\cdot}8081 - 1{\cdot}8466 \\ &= \bar{1}{\cdot}8924 \end{aligned}$$

$$\therefore \quad C = 51^\circ 19' \text{ (from log sin table)}$$

But as $\sin C = \sin (180^\circ - C)$

$C = 180^\circ - 51^\circ 19' = 128^\circ 41'$ will equally satisfy the above value of log sin C.

Thus we are faced with two values of C, namely, $51^\circ 19'$ and $128^\circ 41'$.

If $C = 51^\circ 19'$, $A = 180^\circ - (B + C)$

$$\begin{aligned} &= 180^\circ - (40^\circ + 51^\circ 19') \\ &= 88^\circ 41'. \end{aligned}$$

If $C = 128^\circ 41'$, $A = 180^\circ - (40^\circ + 128^\circ 41') = 11^\circ 19'$.

It might possibly happen that C's larger value when added to the value of B, would in some cases give a greater total than 180°, in which case this larger value of C would be inadmissible.

But in the present case both values of C allow a value for A, and therefore both are possible answers.

How this occurs can be seen in Fig. 63.

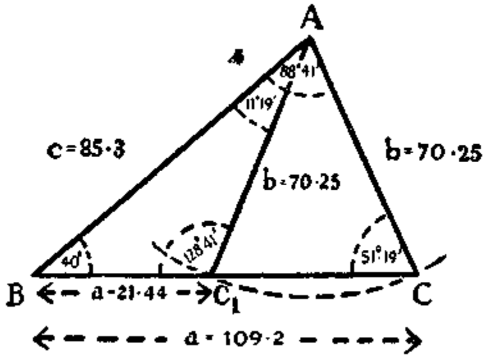

FIG. 63.

If the triangle be drawn as shown, BC is drawn of unlimited length, and $\angle B = 40°$ is drawn. The side c is then measured 85·3 units giving A. With A as centre and radius $b = 70{\cdot}25$ units, an arc is drawn cutting BC in two points C and C_1. These are the two positions possible for C. Thus we see two possible values of C (51° 19′ and 128° 41′) and two corresponding values of A (88° 41′ and 11° 19′).

It only remains to find the two values of a (*i.e.*, BC_1 and BC).

If $$A = 88° 41'$$

$$\frac{a}{\sin 88° 41'} = \frac{70{\cdot}25}{\sin 40}, \text{ giving } a = 109{\cdot}2$$

If $$A = 11° 19'$$

$$\frac{a}{\sin 11° 19'} = \frac{70{\cdot}25}{\sin 40°}, \text{ giving } a = 21{\cdot}44$$

Thus the complete solution is

$$C = 51^\circ\ 19',\ A = 88^\circ\ 41',\ a = 109{\cdot}2$$

and $$C = 128^\circ\ 41',\ A = 11^\circ\ 19',\ a = 21{\cdot}44$$

This is known as the "ambiguous case," and the student is warned to examine carefully for two possible values when evaluating θ from $\sin\theta$ or $\log\sin\theta$. The following diagrams (Fig. 64) will serve to show what possibilities may occur when the given data include two sides and the angle opposite

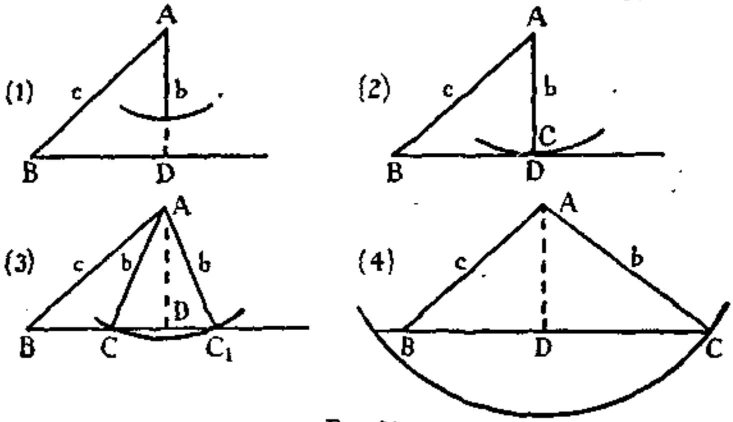

Fig. 64.

one of them, the angle being acute and opposite to the smaller of the two given sides.

The given elements are c, b and B.

A base line is drawn of unlimited length, and at one end, B, is drawn the given angle B. The side BA is then made the given length c. With centre A and radius equal to the given length b, an arc is described as shown. A perpendicular AD is then dropped on the base line

$$AD = c \sin B$$

Case 1.—Fig. 64 (1).

If b is less than AD (*i.e.*, $c \sin B$), the arc does not cut the base line : no triangle is formed and thus there is no solution.

Case 2.—Fig. 64 (2).

If b is equal to AD, the circle will *touch* the base line at D. The triangle is right-angled and only *one* solution is possible.

Case 3.—Fig. 64 (3).

If b is greater than AD, and less than c, the arc will cut the base line in two points, C and C_1, both on the same side of B. Thus there are *two* triangles possible: two solutions, each satisfying the given data, can be found.

Case 4.—Fig. 64 (4).

If b is equal to c, the arc cuts the base in B and in one other point C; if b is greater than c, the arc cuts BD in *two* points on opposite sides of B. In either case there is only *one* triangle satisfying the given data; that is, there is only one solution.

Thus there are two alternative solutions only when b is greater than c sin B, *and less than c.*

3. The Cosine Rule

When the sine rule is not applicable—as, for instance, when two sides and an included angle are given—the following rule, called the *cosine rule,* is used.

Consider the Δ ABC (Fig. 65) with the usual notation.

From A draw AD ($= p$) perpendicular to BC.

Let $\quad DC = n$

and $\quad BD = a - n$

Then, by the Principle of Pythagoras,

$$p^2 = b^2 - n^2$$

and

$$p^2 = c^2 - (a - n)^2$$

$$\therefore \quad b^2 - n^2 = c^2 - (a^2 - 2an + n^2)$$

$$\therefore \quad b^2 - n^2 = c^2 - a^2 + 2an - n^2$$

$$\therefore \quad b^2 = c^2 - a^2 + 2an$$

But $$n = b \cos C$$
$$\therefore \quad b^2 = c^2 - a^2 + 2ab \cos C$$
$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C$$

which implies that, given two sides a and b and the angle C between them, we can find c.

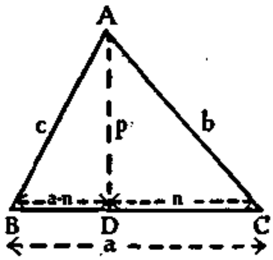

FIG. 65.

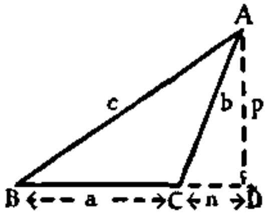

FIG. 66.

If the angle C be obtuse (*Fig.* 66),

$$p^2 = b^2 - n^2$$
$$p^2 = c^2 - (a + n)^2$$
$$\therefore \quad b^2 - n^2 = c^2 - a^2 - 2an - n^2$$
$$\therefore \quad b^2 = c^2 - a^2 - 2an$$

where
$$\begin{aligned} n &= b \cos ACD \\ &= - b \cos ACB \\ &= - b \cos C \end{aligned}$$

$$\therefore \quad b^2 = c^2 - a^2 - 2a(- b \cos C)$$
$$\therefore \quad b^2 = c^2 - a^2 + 2ab \cos C$$
$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C$$

which is identical with the previous formula.

If $C = 90°$, then $\cos C = 0$, in which case $c^2 = a^2 + b^2$. Thus, in all cases,

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C}$$

Using this notation, we ma say,

If two variable quantities x and y are so related that y is a function of x, then if x receives an increment Δx, y will receive a corresponding increment Δy.

Thus it follows that if in an expression which expresses y as a function of x, an increment Δx is given to x, then x must be replaced by $(x + \Delta x)$ throughout, and y by $(y + \Delta y)$.

Thus if $y = x^2 + 5x - 2$ and x receives an increment Δx, then y will receive an increment Δy, and the equation becomes $y + \Delta y = (x + \Delta x)^2 + 5(x + \Delta x) - 2$.

Similarly, if the distance s passed over by a moving body in time t be expressed by $s = 16t + 4t^2$ and t and receive increments Δt and Δs respectively, then

$$s + \Delta s = 16(t + \Delta t) + 4(t + \Delta t)^2$$

6. Gradient and Slope

Consider any two points P and Q on a straight line (Fig. 90).

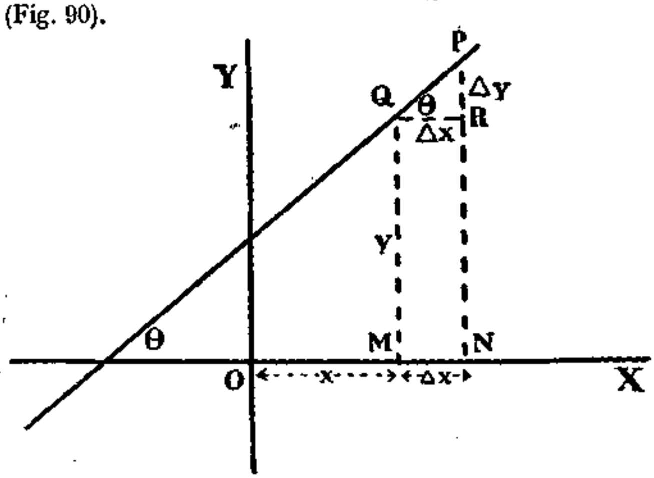

FIG. 90.

Taking logs. of both sides,

$$\begin{aligned}\log \sin B &= \log 3 + \log \sin 50^\circ - \log 3{\cdot}836\\ &= 0{\cdot}4771 + \bar{1}{\cdot}8843 - 0{\cdot}5839\\ &= 0{\cdot}3614 - 0{\cdot}5839\\ &= \bar{1}{\cdot}7775\end{aligned}$$

$$\therefore \quad \sin B = 0{\cdot}5991$$

$$\therefore \quad B = 36^\circ\,48' \text{ or } (180^\circ - 36^\circ\,48') = 143^\circ\,12'$$

If $B = 36^\circ\,48'$ and $C = 50^\circ$, $A = 180^\circ - 86^\circ\,48' = 93^\circ\,12'$.

If $B = 143^\circ\,12'$ and $C = 50^\circ$, then $(B + C) > 180^\circ$, which is impossible.

$\therefore$ Complete solution is :

$$c = 3{\cdot}836 \textit{ ins.},\ B = 36^\circ\,48',\ A = 93^\circ\,12'.$$

(2) *Find the angles of the triangle whose sides are* 48, 38, *and* 52 *chains respectively.*

Let $a = 48$, $b = 38$ and $c = 52$ chains.

Then

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc}\\ &= \frac{38^2 + 52^2 - 48^2}{2 \times 38 \times 52}\\ &= \frac{1444 + 2704 - 2304}{2 \times 38 \times 52}\\ &= \frac{1844}{2 \times 38 \times 52}\end{aligned}$$

Taking logs. of both sides,

$$\begin{aligned}\log \cos A &= \log 1844 - (\log 2 + \log 38 + \log 52)\\ &= 3{\cdot}2657 - (0{\cdot}3010 + 1{\cdot}5798 + 1{\cdot}7160)\\ &= \bar{1}{\cdot}6689\\ &= \log \cos 62^\circ\,11'\end{aligned}$$

$$\therefore \quad A = 62^\circ\,11'$$

Now apply the Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\therefore \quad \frac{48}{\sin 62^\circ 11'} = \frac{38}{\sin B}$$

$$\therefore \quad \sin B = \frac{38 \times \sin 62^\circ 11'}{48}$$

$$\begin{aligned} \therefore \quad \log \sin B &= \log 38 + \log \sin 62^\circ 11' - \log 48 \\ &= 1\cdot5798 + \bar{1}\cdot9466 - 1\cdot6812 \\ &= \bar{1}\cdot8452 \\ &= \log \sin 44^\circ 26' \end{aligned}$$

$$\therefore \quad B = 44^\circ 26'$$

(or $B = 180^\circ - 44^\circ 26' = 135^\circ 34'$, which will be seen to be inadmissible with $A = 62^\circ 11'$).

$\therefore \quad A = 62^\circ 11'$, $B = 44^\circ 26'$

and $C = 180^\circ - (62^\circ 11' + 44^\circ 26')$

$$\therefore \quad A = \mathit{62^\circ\ 11'},\ B = \mathit{44^\circ\ 26'},\ C = \mathit{73^\circ\ 23'}.$$

EXERCISE 14

Solve the following triangles:

1. $A = 88^\circ$, $B = 36^\circ$, $a = 9\cdot5$ ins.
2. $A = 60^\circ$, $b = 4\cdot75$ ins., $c = 5$ ins.
3. $a = 27$ yds., $c = 21$ yds., $B = 60^\circ$.
4. $c = 220$ yds., $b = 190$ yds., $C = 60^\circ$.
5. $a = 38$ ft., $b = 25$ ft., $C = 78^\circ$.
6. $a = 5\cdot2$ ins., $b = 5\cdot7$ ins., $c = 9\cdot4$ ins.
7. $A = 50^\circ$, $C = 66^\circ$, $a = 500$ yds.
8. Find B and C if $A = 50^\circ$, $b = 238$ yds., $a = 194$ yds.
9. If $a = 96$ ft., $c = 100$ ft., and $C = 66^\circ$, show that the only value of A is $61^\circ 18'$ and of B, $52^\circ 42'$.
10. If $a = 40\cdot9$, $b = 38\cdot5$, $c = 38\cdot1$, find the largest angle.
11. If the angles of a triangle are 50°, 60° and 70°, and the area is 100 sq. ins., find the shortest side.

12. If $a = 6{\cdot}2$ ins., $b = 7{\cdot}8$ ins., and $C = 52°$, find the area of the triangle.

13. Show that the area of a parallelogram is equal to half the product of its diagonals and the sine of the angle included between them.

14. Two aeroplanes start simultaneously from the same aerodrome and fly in directions making 60° with each other. If one flies at the rate of 100 m.p.h., find the speed of the other if they are 350 mls. apart at the end of $2\frac{1}{2}$ hrs.

15. The length of the side BC of the triangle ABC is 14·5 ins., $A\hat{B}C = 71°$, $B\hat{A}C = 57°$. Calculate the lengths of the sides AC and AB. (N.C.T.E.C., 1934.)

16. A rod of length 9·1 ft. slides with its ends A and B, one along each of two straight lines, meeting at an angle in O. In one of the positions of the rod, A and B are 8·5 ft. and 9·6 ft. respectively from O. Calculate the angle between the guides and check your result by a scale drawing. (N.C.T.E.C., 1935.)

17. Two forces of 3·5 tons and 6·7 tons act together in the same plane at a point. Both forces pull away from the point, and the angle between the directions of the forces is 75°. Represent these forces on a diagram and complete the parallelogram of which they are adjacent sides. The single force equivalent to the above forces is represented by the diagonal of the parallelogram drawn through the meeting point of the forces.

Calculate:

(i) The magnitude in tons of this single force.

(ii) The angle, to the nearest degree, which this single force makes with the line of action of the greater of the above forces.

N.B.—Readings taken from a scale drawing will not be accepted as a solution. (U.E.I., 1932.)

18. The triangle CAB is the triangle of velocities for steam entering a turbine blade. CA represents 1500 ft. per sec., AB 600 ft. per sec., and the angle CAB is 25°.

Calculate:

(i) CB in ft. per sec.

(ii) The angle CBD formed between CB and AB produced. (U.E.I., 1935.)

19. The lengths of the sides of a triangle are 4, 6 and 8 ins. Find the area of the triangle and obtain the value of the largest angle to the nearest degree. (U.L.C.I., 1936.)

20. (*a*) Write down the formulæ you would use to solve a triangle having given (i) one side and two angles, (ii) three sides.

(*b*) In a quadrilateral ABCD, AB = 3 ins., BC = 4 ins., CD = 7·4 ins., DA = 4·4 ins., and the angle ABC is 90°.

Determine the angle ADC in degrees. (U.L.C.I., 1935.)

21. Two ships leave port at the same time. The first steams S.E. at 18 m.p.h., and the second 25° W. of S. at 15 m.p.h. Calculate, by trigonometry, the time that will have elapsed when they are 86 m. apart.

(N.C.T.E.C., 1933.)

22. The lengths of the sides of a triangle are 5·6 ins., 5 ins., and 3·4 ins. Calculate the lengths of the two parts into which the longest side is divided by the perpendicular drawn to it from the opposite vertex, and thence calculate the angles of the triangle. (N.C.T.E.C., 1931.)

23. Construct a triangle ABC whose base AB is 5 ins. long, the angle BAC = 55° and the angle ABC = 48°. Calculate the lengths of the sides AC and BC and the area of the triangle.

Compare the calculated lengths with the measured lengths of the sides. (U.L.C.I., 1927.)

CHAPTER 10

THE ADDITION FORMULÆ

1. It is sometimes expedient to express the trigonometrical ratios of a compound angle in terms of the ratios of its component angles; for instance, sin (A + B) may be conveniently expressed in terms of sin A and sin B in the course of a calculation.

2. To Express sin (A + B) in Terms of the Ratios of A and B

Draw a vertical line LN (Fig. 67).

At L, construct angle NLM equal to A, and angle NLP equal to B. Through N, draw MNP perpendicular to LN.

Let the lengths of the various lines be denoted by small letters as shown.

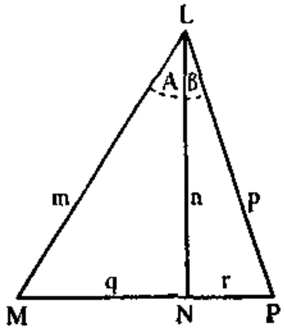

FIG. 67.

Then, area of $\triangle$LMP = area of $\triangle$LMN + area of $\triangle$LPN. But the area of a triangle = $\frac{1}{2}$ (product of two sides and the sine of the angle contained between them).

$$\therefore \quad \text{Area of } \triangle LMP = \tfrac{1}{2}mp \sin (A + B)$$
$$\text{,,} \quad \text{,,} \ \triangle LMN = \tfrac{1}{2}mn \sin A$$
$$\text{,,} \quad \text{,,} \ \triangle LPN = \tfrac{1}{2}pn \sin B$$
$$\therefore \qquad \tfrac{1}{2}mp \sin (A + B) = \tfrac{1}{2}mn \sin A + \tfrac{1}{2}pn \sin B.$$

Dividing through by $\frac{1}{2}mp$ {*i.e.*, the coefficient of sin (A + B)},

$$\sin(A + B) = \frac{n}{p}\sin A + \frac{n}{m}\sin B$$

But $\qquad \frac{n}{p} = \cos B$, and $\frac{n}{m} = \cos A$.

$$\therefore \quad \sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

3. To Express cos (A + B) in Terms of the Ratios of A and B

Using the same figure, and applying the Cosine Rule,

$$MP^2 = LM^2 + LP^2 - 2LM \cdot LP \cdot \cos MLP$$

$$\therefore \quad (q + r)^2 = m^2 + p^2 - 2mp\cos(A + B)$$

$$\therefore \quad \cos(A + B) = \frac{m^2 + p^2 - (q + r)^2}{2mp}$$

$$= \frac{m^2 + p^2 - q^2 - 2qr - r^2}{2mp}$$

$$= \frac{(m^2 - q^2) + (p^2 - r^2) - 2qr}{2mp}$$

But by the Principle of Pythagoras, $m^2 - q^2 = n^2$ and $p^2 - r^2 = n^2$.

$$\therefore \quad \cos(A + B) = \frac{2n^2 - 2qr}{2mp}$$

$$= \frac{n^2 - qr}{mp}$$

$$= \frac{n^2}{mp} - \frac{qr}{mp}$$

$$= \frac{n}{m} \cdot \frac{n}{p} - \frac{q}{m} \cdot \frac{r}{p}$$

$$\therefore \quad \cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B.$$

4. To Express sin (A − B) in Terms of the Ratios of A and B

(A being greater than B)

Draw a vertical line LN (Fig. 68) with NM of unlimited length at right angles to it at N.

Construct angles NLM and NLP equal to A and B, respectively. Let the length of the lines be indicated by the small letters as shown.

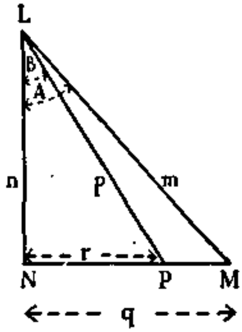

FIG. 68.

Then,

Area of $\triangle$LPM = area of $\triangle$LNM − area of $\triangle$LNP

$\therefore \quad \frac{1}{2}pm \sin (A - B) = \frac{1}{2}nm \sin A - \frac{1}{2}np \sin B$

Dividing through by $\frac{1}{2}pm$ {*i.e.*, coefficient of sin (A − B)}

$$\sin (A - B) = \frac{n}{p} \sin A - \frac{n}{m} \sin B$$

$$\therefore \quad \textit{sin}\,(A - B) = \textit{sin}\,A\,\textit{cos}\,B - \textit{cos}\,A\,\textit{sin}\,B.$$

5. To Express cos (A − B) in Terms of the Ratios of A and B

(A being greater than B)

Using the same figure, and applying the Cosine Rule,

$$PM^2 = LP^2 + LM^2 - 2LP \cdot LM \cdot \cos PLM$$

$$\therefore \quad (q - r)^2 = p^2 + m^2 - 2pm \cos (A - B)$$

$$\therefore \cos (A - B) = \frac{p^2 + m^2 - (q - r)^2}{2pm}$$

$$= \frac{p^2 + m^2 - q^2 + 2qr - r^2}{2pm}$$

$$= \frac{(p^2 - r^2) + (m^2 - q^2) + 2qr}{2pm}$$

$$= \frac{n^2 + n^2 + 2qr}{2pm}$$

$$= \frac{n^2 + qr}{pm}$$

$$= \frac{n^2}{pm} + \frac{qr}{pm}$$

$$= \frac{n}{p} \cdot \frac{n}{m} + \frac{q}{m} \cdot \frac{r}{p}$$

$$\therefore \quad \cos(A - B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B$$

6. We thus obtain the following :

$$\mathbf{\sin(A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B}$$
$$\mathbf{\sin(A - B) = \sin A \,.\, \cos B - \cos A \,.\, \sin B}$$
$$\mathbf{\cos(A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B}$$
$$\mathbf{\cos(A - B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B}$$

Although these have been proved only for acute angles, they are equally true for angles of all sizes; more general proofs are omitted only because, at this stage, it is the application of these identities which is more important.

The corresponding tangent formulæ can be obtained as exercises on the above.

7. To Express tan (A + B) and tan (A − B) in Terms of tan A and tan B

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin A \,.\, \cos B + \cos A \,.\, \sin B}{\cos A \,.\, \cos B - \sin A \,.\, \sin B}$$

Now divide numerator and denominator each by $\cos A \,.\, \cos B$.

$$\tan(A + B) = \frac{\dfrac{\sin A \,.\, \cos B}{\cos A \,.\, \cos B} + \dfrac{\cos A \,.\, \sin B}{\cos A \,.\, \cos B}}{\dfrac{\cos A \,.\, \cos B}{\cos A \,.\, \cos B} - \dfrac{\sin A \,.\, \sin B}{\cos A \,.\, \cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$

$$\therefore \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

and $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}.$$

The latter will easily be obtained from the first by the change of sign.

Examples

(1) *If* $\sin A = 0{\cdot}5$ *and* $\cos B = 0{\cdot}3$, *find the value of* $\sin(A + B)$ *and of* $\cos(A - B)$ *without evaluating the angles* A *and* B.

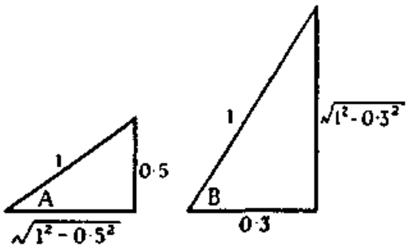

FIG. 69.

Since

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

and $$\cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

we must first find the values of cos A and sin B.

Drawing a Δ in which A represents the angle, and with hypotenuse and perpendicular 1 and 0·5 as shown, Fig. 69, we see that the length of the base is $\sqrt{1^2 - 0{\cdot}5^2} = 0{\cdot}866$

$$\therefore \quad \cos A = 0{\cdot}866$$

Similarly,

$$\sin B = \sqrt{1^2 - 0{\cdot}3^2} = 0{\cdot}9539$$

$$\therefore \sin(A + B) = (0{\cdot}5 \times 0{\cdot}3) + (0{\cdot}866 \times 0{\cdot}9539)$$
$$= 0{\cdot}15 + 0{\cdot}8260 = \mathit{0{\cdot}9760}$$

and $$\cos(A - B) = (0{\cdot}866 \times 0{\cdot}3) + (0{\cdot}5 \times 0{\cdot}9539)$$
$$= 0{\cdot}2598 + 0{\cdot}4770 = \mathit{0{\cdot}7368}$$

(2) *Given that tan* A = 0·7 *and tan* (A − B) = 0·3, *find tan* B.

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$\therefore \quad 0{\cdot}3 = \frac{0{\cdot}7 - \tan B}{1 + 0{\cdot}7 \tan B}$$

$$\therefore \quad (0{\cdot}3)(1 + 0{\cdot}7 \tan B) = 0{\cdot}7 - \tan B$$

$$\therefore \quad 0{\cdot}3 + 0{\cdot}21 \tan B = 0{\cdot}7 - \tan B$$

$$\therefore \quad \tan B + 0{\cdot}21 \tan B = 0{\cdot}7 - 0{\cdot}3$$

$$\therefore \quad 1{\cdot}21 \tan B = 0{\cdot}4$$

$$\therefore \quad \tan B = \frac{0{\cdot}4}{1{\cdot}21}$$

$$\therefore \quad \tan B = 0{\cdot}3306.$$

8. A very useful application of these identities is often found in electrical and mechanical problems, where it is sometimes necessary to change an expression of such a form as

$$a \sin pt + b \cos pt$$

into the form M sin $(pt + c)$, where a, b, and p are constants and c is an angle.

Since

$$\begin{aligned} M \sin(pt + c) &= M(\sin pt \cos c + \cos pt \sin c) \\ &= M \sin pt \cos c + M \cos pt \sin c \end{aligned}$$

if $a \sin pt + b \cos pt$ is to be identical with M (sin $pt + c$) then $M \sin pt \cos c + M \cos pt \sin c$ must be identical with

$$a \sin pt + b \cos pt$$

∴ The coefficients of sin pt in both expressions must be equal, and similarly the coefficients of cos pt.

$$\therefore \qquad M \cos c = a$$

and $$M \sin c = b$$

By division $$\frac{M \sin c}{M \cos c} = \tan c = \frac{b}{a}$$

By squaring and adding,

$$M^2 (\cos^2 c + \sin^2 c) = a^2 + b^2$$

$$\therefore \quad M = \sqrt{a^2 + b^2} \text{ (since } \cos^2 c + \sin^2 c = 1)$$

$$\therefore \quad a \sin pt + b \cos pt = M \sin (pt + c)$$

$$\textit{where } M = \sqrt{a^2 + b^2} \textit{ and } \tan c = \frac{b}{a}.$$

Examples

(1) *Change* $9{\cdot}2 \sin 3t + 8{\cdot}4 \cos 3t$ *into the form* $M \sin (3t + c)$.

Here
$$M = \sqrt{9{\cdot}2^2 + 8{\cdot}4^2}$$
$$= \sqrt{155{\cdot}2}$$
$$= 12{\cdot}46$$

Also

$$\tan c = \frac{b}{a}, \text{ where } b = 8{\cdot}4 \text{ and } a = 9{\cdot}2$$

$$= \frac{8{\cdot}4}{9{\cdot}2} = 0{\cdot}9131$$

$$\therefore \quad c \doteq 42^\circ\ 24' \text{ or (as is more usual) } 0{\cdot}74 \text{ radian}$$

$$\therefore \quad 9{\cdot}2 \sin 3t + 8{\cdot}4 \cos 3t = \mathit{12{\cdot}46} \sin (3t + 0{\cdot}74)$$

(2) *Express* $5 \sin 2\pi ft - 3 \cos 2\pi ft$ *in the form* $A \sin (2\pi ft + g)$.

Here $\quad A = \sqrt{5^2 + (-3)^2} = 5{\cdot}831$

and $\quad \tan g = -\frac{3}{5}$

$$\therefore g = -30^\circ\ 58' = -0{\cdot}5405 \text{ radian (using smallest value)}$$

$$\therefore \quad 5 \sin 2\pi ft - 3 \cos 2\pi ft = 5{\cdot}831 \sin (2\pi ft - 0{\cdot}5405).$$

Ratios of Multiple and Sub-multiple Angles

9. Perhaps the most useful applications of the previous "Addition Formulæ" occur when $A = B$.

Then

$$\sin (A + B) = \sin A \, . \cos B + \cos A \, . \sin B \text{ becomes}$$

$$\sin 2A = \sin A \, . \cos A + \cos A \, . \sin A$$

$$\therefore \quad \mathbf{\sin 2A = 2 \sin A \,.\, \cos A}$$

Similarly,

$$\cos (A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B \quad \text{becomes}$$

$$\cos 2A = \cos A \,.\, \cos A - \sin A \,.\, \sin A$$

$$\therefore \quad \mathbf{\cos 2A = \cos^2 A - \sin^2 A}$$

Also,

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B} \quad \text{becomes}$$

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \,.\, \tan A}$$

$$\therefore \quad \mathbf{\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}}$$

An important development of $\cos 2A$ is met with, when its value, $\cos^2 A - \sin^2 A$, is coupled with

$$\cos^2 A + \sin^2 A = 1$$

For

$$\cos 2A = \cos^2 A - \sin^2 A$$

and

$$1 = \cos^2 A + \sin^2 A$$

which, by addition, give

$$\cos 2A + 1 = 2 \cos^2 A$$

$$\therefore \quad \cos 2A = 2 \cos^2 A - 1$$

and, by subtraction,

$$\cos 2A - 1 = - 2 \sin^2 A$$

$$\therefore \quad \cos 2A = 1 - 2 \sin^2 A$$

If $2A$ be written as θ,

then

$$A = \frac{\theta}{2}$$

$$\therefore \quad \sin \theta = 2 \sin \frac{\theta}{2} \,.\, \cos \frac{\theta}{2}$$

$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$$

$$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

by substitution in the previous formulæ.

Examples

(1) *If* $\cos$ B $= 0{\cdot}8$, *find* $\sin$ 2B, $\cos$ 2B *and* $\tan$ 2B *without evaluating* B.

Since $\cos B = \frac{4}{5}$

$\sin B = \frac{3}{5}$ (see Fig. 70)

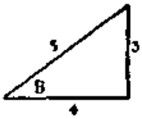

Fig. 70.

$$\therefore \quad \sin 2B = 2 \sin B \,.\, \cos B$$
$$= 2 \times \tfrac{3}{5} \times \tfrac{4}{5} = \tfrac{24}{25}$$
$$= \mathit{0{\cdot}96}$$

$$\cos 2B = \cos^2 B - \sin^2 B$$
$$= \tfrac{16}{25} - \tfrac{9}{25} = \tfrac{7}{25}$$
$$= \mathit{0{\cdot}28}$$

$$\tan 2B = \frac{\sin 2B}{\cos 2B} = \frac{0{\cdot}96}{0{\cdot}28}$$
$$= \mathit{3{\cdot}428}$$

(2) *Prove that* $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$

$$\frac{\sin 2A}{1 + \cos 2A} = \frac{2 \sin A \,.\, \cos A}{1 + (2 \cos^2 A - 1)}$$
$$= \frac{2 \,.\, \sin A \,.\, \cos A}{2 \cos^2 A}$$
$$= \frac{\sin A}{\cos A} = \tan A.$$

(3) *Prove that* $\dfrac{\cos 2\theta}{1 + \sin 2\theta} = \dfrac{1 - \tan \theta}{1 + \tan \theta}$.

$$\frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\cos^2 \theta - \sin^2 \theta}{1 + 2 \sin \theta \,.\, \cos \theta}$$
$$= \frac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \,.\, \cos \theta}$$
$$= \frac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{(\sin \theta + \cos \theta)(\sin \theta + \cos \theta)}$$
$$= \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta}$$

$$= \frac{1 - \frac{\sin \theta}{\cos \theta}}{1 + \frac{\sin \theta}{\cos \theta}} \text{ (by dividing each term by } \cos \theta)$$

$$= \frac{1 - \tan \theta}{1 + \tan \theta}.$$

Trigonometrical Equations

10. Sometimes equations are met with containing one or more trigonometric functions, and it is required to find the angle involved. In such cases it may be necessary to use our knowledge of the relations between different trigonometric functions of the angle or angles, so as to reduce the number of different ratios before we can solve for the angle.

Examples

(1) *Given* $5 \sin^2 \theta - 3 \cos \theta = 2{\cdot}25$, *find all positive values of* θ *up to* $360°$.

Here there are involved two different ratios (sin and cos), which gives the equation the appearance of a simultaneous equation in $\sin \theta$ and $\cos \theta$. But since $\sin^2 \theta = 1 - \cos^2 \theta$, we can reduce the equation to a quadratic in $\cos \theta$, and so solve it

$$5 \sin^2 \theta - 3 \cos \theta = 2{\cdot}25$$
$$5(1 - \cos^2 \theta) - 3 \cos \theta = 2{\cdot}25$$
$$5 - 5 \cos^2 \theta - 3 \cos \theta = 2{\cdot}25$$
$$\therefore \quad 5 \cos^2 \theta + 3 \cos \theta - 2{\cdot}75 = 0$$

$$\therefore \quad \cos \theta = \frac{-3 \pm \sqrt{9 + 55}}{10}$$
$$= \frac{-3 \pm 8}{10}$$
$$= -\tfrac{11}{10} \text{ or } \tfrac{5}{10}$$
$$= -\tfrac{11}{10} \text{ or } \tfrac{1}{2}$$

But it is the values of θ which are required, and not merely those of $\cos\theta$. Therefore, we must find all values of θ up to 360°, such that $\cos\theta = -\frac{11}{10}$ or $\frac{1}{2}$. Since no angle has a cosine value greater than 1, or less than -1, $\cos\theta = -\frac{11}{10}$ is inadmissible.

If $\cos\theta = \frac{1}{2}$, $\theta = 60°$ or the corresponding angle in any quadrant where the cosine value is positive.

This is found only in the fourth quadrant where $\cos(360° - \theta) = \cos\theta$ (see p. 144).

$$\therefore \quad \theta = 60° \text{ or } 300°.$$

(2) *Solve* $\cos 2\theta = 1 - 6\cos^2\theta$ *for all positive values of* θ *up to* 360°.

Substituting $2\cos^2\theta - 1$ (see p. 180, § 9) for $\cos 2\theta$,

Then $\quad 2\cos^2\theta - 1 = 1 - 6\cos^2\theta$

$$\therefore \quad 8\cos^2\theta = 2$$

$$\therefore \quad \cos^2\theta = \tfrac{1}{4}$$

$$\therefore \quad \cos\theta = \pm\tfrac{1}{2}$$

If $\quad \cos\theta = +\frac{1}{2}$, $\theta = 60°$ or 300°.

If $\quad \cos\theta = -\frac{1}{2}$, $\theta = 120°$ or 240°.

$$\therefore \quad \theta = 60°,\ 120°,\ 240° \text{ or } 300°.$$

EXERCISE 15

In Nos. 1–10, the angle must not be evaluated.

1. If $\sin A = 0{\cdot}52$, and $\cos B = 0{\cdot}78$, find the values of $\sin(A + B)$ and $\cos(A + B)$.
2. If $\sin A = 0{\cdot}72$ and $\cos B = 0{\cdot}91$, find $\cos(A - B)$.
3. If $\sin B = 0{\cdot}23$ and $\cos A = 0{\cdot}309$, find $\sin(A - B)$.
4. If $\cos B = 0{\cdot}32$, and $\sin A = 0{\cdot}71$, find $\sin(A + B)$ and $\tan(A + B)$.
5. If $\tan A = 1{\cdot}2$ and $\tan B = 0{\cdot}4$, find $\tan(A + B)$ and $\tan(A - B)$.
6. If $\sin A = \frac{60}{61}$ and $\cos B = \frac{40}{41}$, find $\cos(A - B)$.

7. If $\tan A = 1{\cdot}782$ and $\tan (A - B) = 0{\cdot}7054$, find $\tan B$.

8. Given that $\sin A = \frac{3}{5}$, find $\sin 2A$, $\cos 2A$ and $\tan 2A$.

9. If $\cos B = 0{\cdot}66$, find $\sin 2B$ and $\cos 2B$.

10. Find $\sin A$, and $\sin \frac{A}{2}$, if $\sin 2A = 0{\cdot}97$.

11. Express $3{\cdot}6 \cos 6t + 2{\cdot}7 \sin 6t$ in the form $M \sin (6t + c)$.

12. Express $30 \sin (2\pi nt) + 25 \cos (2\pi nt)$ in the form $A \sin (2\pi nt + g)$.

13. Express $5 \sin 4t - 3 \cos 4t$ in the form $A \sin (4t + g)$, using the smallest possible value of g in radians.

Prove the following identities for all values of the given angle.

14. $\cos (A + B) + \cos (A - B) = 2 \cos A \, . \cos B$.

15. $\sin (A + B) - \sin (A - B) = 2 \cos A \, . \sin B$.

16. $\dfrac{\sin (A + B)}{\cos A \, . \cos B} = \tan A + \tan B$.

17. $\cot A + \tan B = \dfrac{\cos (A - B)}{\sin A \, . \cos B}$.

18. $\cos (A + 45°) + \sin (A - 45°) = 0$.

19. $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2 \dfrac{x}{2}$.

20. $\dfrac{2 \operatorname{cosec} 2\theta}{\operatorname{cosec} \theta} = \sec \theta$.

21. $\dfrac{2 \tan A}{1 + \tan^2 A} = \sin 2A$.

22. $\dfrac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \sec 2\theta$.

23. $\dfrac{\sin A}{1 - \cos A} = \cot \dfrac{A}{2}$.

24. $\dfrac{1 + \sec \theta}{\sec \theta} = 2 \cos^2 \dfrac{\theta}{2}$.

25. $\dfrac{\sin^2 2A}{2 \cos^2 A} = 1 - \cos 2A$.

26. $\cos^4 B - \sin^4 B = \cos 2B$.

27. $\dfrac{\sin 4A}{\sin 2A} = 2 \cos 2A.$

28. $\dfrac{\cos (A - 45°)}{\cos (A + 45°)} = \dfrac{1 + \sin 2A}{\cos 2A}.$

29. $\sin 3A = 3 \sin A - 4 \sin^3 A.$

(Start with $\sin 3A = \sin (2A + A)$.)

30. $\cos 3A = 4 \cos^3 A - 3 \cos A.$

31. $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$

Solve the following equations for all values of the angle from 0° to 360°.

32. $6 \sin \theta = \tan \theta.$
33. $2 \tan^2 \theta - 3 \tan \theta + 1 = 0.$
34. $\cos \theta + \tan \theta = \sec \theta.$
35. $5 \tan^2 x - \sec^2 x = 11.$
36. $4 \cos \theta = 3 \tan \theta.$
37. $4 \sin^2 \theta - 3 \cos \theta = 1{\cdot}5.$
38. $4 \sin^2 \theta = \cos^2 \theta.$
39. $3 \cos^2 A + 5 \sin^2 A = 4.$
40. $4 \sin^2 x + 5 \cos^2 x = 4{\cdot}25.$
41. $\cos \theta - \sin \theta = 0{\cdot}8.$
42. $4 \cos \theta = 3 \sec \theta.$
43. $\tan x \,.\, \operatorname{cosec} x = 5.$
44. $\cot^2 x + \operatorname{cosec}^2 x = 3.$
45. $\sin \theta + \sin 2\theta = 0.$
46. $\sin 2\theta = \sin \theta.$
47. $\cos \theta = \cos 2\theta.$
48. $\cos 2\theta - \cos \theta = 2.$

CHAPTER 11

THE PLOTTING OF MORE DIFFICULT GRAPHS

1. Recapitulation

In the First-Year Course, the graphs of $y = ax + b$, $y = ax^2 + bx + c$ and $y = \frac{a}{x}$ were dealt with.

It will be remembered that $y = ax + b$ represents straight lines which differ in slope and position according to the values borne by a and b. The value of a denotes the gradient of the line, and the value of b the intercept on the y axis.

The graph of $y = ax^2 + bx + c$ was found to be a curve with one turning-point (a parabola). A reference to Chapter 3 of this volume will serve as a useful reminder.

The graph of $y = \frac{a}{x}$ was found to be a curve of the hyperbola type.

In this chapter it is proposed to deal with a few graphs of harder types.

2. Graphs of Cubic Expressions

A cubic expression is one in which the highest power of the variable is the third: x^3, $3x^3 + 7$, $5x^3 - 2x + 3$, and $7 + 2x - 5x^2 - 4x^3$ are cubic expressions, or cubic functions of x.

3. The Graph of $y = x^3$

Connected values of x and y are tabulated below:

x . . .	-4	-3	-2	-1	0	1	2	3	4
$y = x^3$ . .	-64	-27	-8	-1	0	1	8	27	64

An examination of these values reveals that:

(*a*) When x is negative, y is negative.

(*b*) When x is zero, y is zero.

(*c*) When x is positive, y is positive.

(*d*) As x increases, y increases.

(*e*) When x is infinitely large (positive or negative) y is infinitely large (positive or negative respectively).

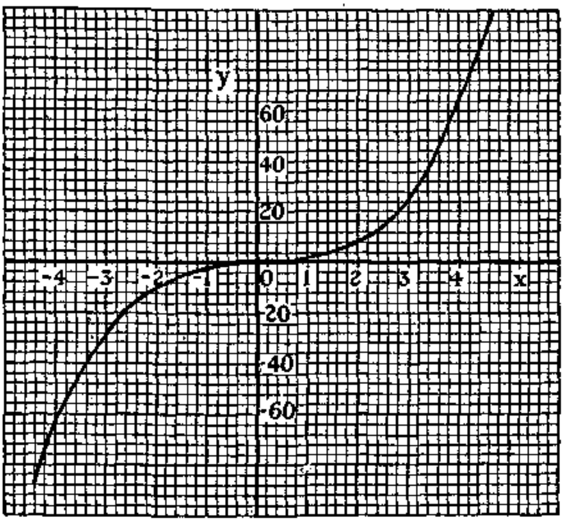

FIG. 71. Graph of $y = x^3$.

(*f*) For each numerically equal pair of positive and negative values of x, there are numerically equal pairs of positive and negative values of y; there are two symmetrical (but relatively reversed) portions of the curve.

On plotting the curve (Fig. 71), the above conclusions are verified. We also note that the origin, O, is the *centre of*

symmetry : here the curve changes from concave downwards to concave upwards, the x axis being tangential to both portions; for this reason, O is called a *Point of Inflection.*

4. The Graph of $y = -x^3$

Tabulating and plotting as before, we obtain :

x . .	-4	-3	-2	-1	0	1	2	3	4
$y = -x^3$	$+64$	$+27$	$+8$	$+1$	0	-1	-8	-27	-64

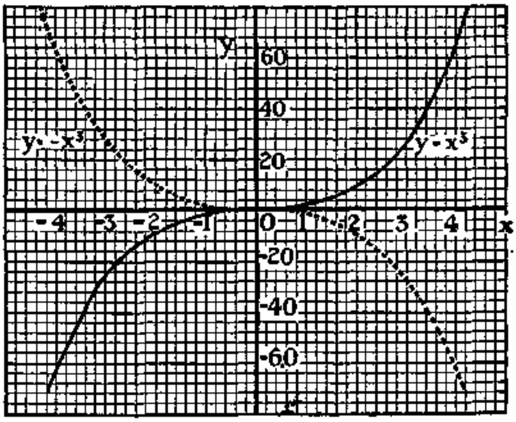

FIG. 72.—Graphs of $y = x^3$ and $y = -x^3$.

The graphs of $y = x^3$ and $y = -x^3$ may be looked upon as " reflections " of each other.

5. The Graph of $y = ax^3$ (where a is constant)

The student will easily see that

(*a*) The introduction of the constant a simply multiplies the y value by a, for each value of x.

(b) When $x = 0, y = 0$.

(c) When $x \longrightarrow \infty, y \longrightarrow \infty$.

(d) In general form, the graph is similar to that of $y = x^3$.

(e) When a is negative, the graph will be similar to that of $y = -x^3$.

6. The Graph of $y = ax^3 + b$

The effect of adding the constant b (which may be positive or negative) is to increase or decrease the value of y by the value of b. The graph will be the same shape as the graph of $y = ax^3$; but its position will be different, since it no longer passes through the origin, for when $x = 0, y = b$.

The student will now be able to visualise graphs such as $10 + x^3$, $7 - x^3$, $2x^3 - 8$, etc.

7. The Graph of $y = ax^3 + bx + c$

Here we find the variable in the first as well as the third degree.

Let $a = 3$, $b = 5$ and $c = 10$.

The graph of $y = 3x^3 + 5x + 10$ can be drawn by tabulating and plotting in the usual way.

x	-3	-2	-1	0	1	2	3
$3x^3$	-81	-24	-3	0	3	24	81
$5x$	-15	-10	-5	0	5	10	15
10	10	10	10	10	10	10	10
$y = 3x^3 + 5x + 10$.	-86	-24	2	10	18	44	106

Here we notice that as x increases from -3 through 0 to $+3$, y increases from -86 to $+106$, passing through the value 0 when x is between -2 and -1 (probably nearer to -1 than -2).

From the graph (Fig. 73) it is seen that $y = 0$ only when $x = -1{\cdot}1$ (approx.).

The graph is a cubic curve similar to $y = x^3$, having a point of inflection, but no turning points.

We also note that the equation $3x^3 + 5x + 10 = 0$ has one real root which is $x = -1{\cdot}1$ (approx.).

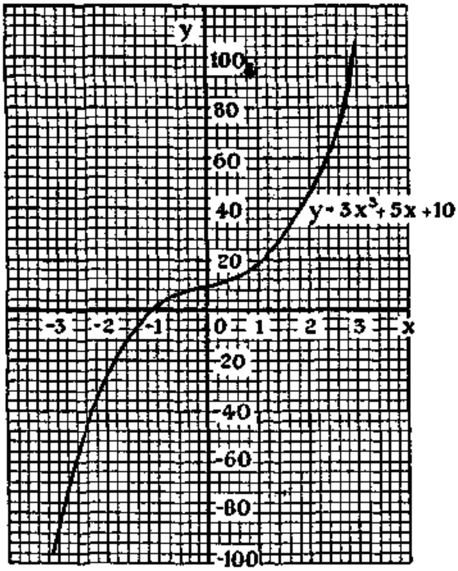

FIG. 73.—Graph of $y = 3x^3 + 5x + 10$.

If a more accurate value of this root should be required, it would be necessary to plot the graph from $x = -1{\cdot}5$ to $x = -0{\cdot}5$ on a large scale.

We could also use this graph to solve the equation

$$3x^3 + 5x + 10 = 50$$

by drawing a horizontal line through $y = 50$ and reading

off the x value of the point of intersection. It appears to be 2·15 approximately.

Since $3x^3 + 5x + 10 = 50$ can be written as

$$3x^3 + 5x - 40 = 0,$$

the real root of this equation is also 2·15.

8. Alternative Method of Solving $3x^3 + 5x + 10 = 0$

Since $3x^3 + 5x + 10 = 0$ may be written as

$$3x^3 = -5x - 10,$$

we may proceed to draw the graphs of

$$y = 3x^3 \text{ and } y = -5x - 10,$$

and then find their point of intersection.

$y = 3x^3$ may be plotted by using the values of $3x^3$ shown

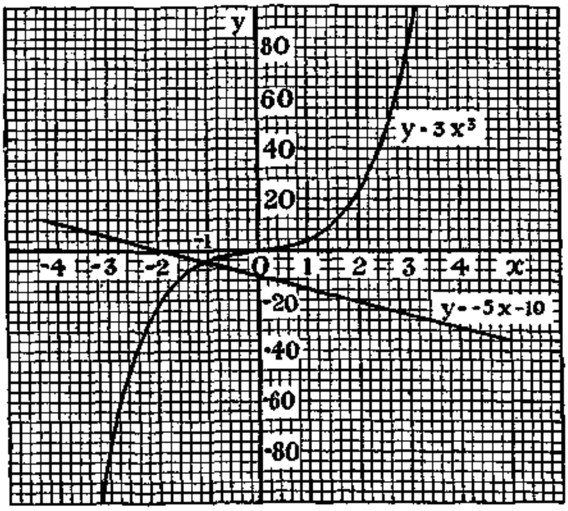

FIG. 74.—Graph showing intersection of $y = 3x^3$ and $y = -5x - 10$.

in the table used for the last graph and since $y = -5x - 10$ is a straight line, we need only two pairs of corresponding values of x and y, such as:

x	0	4
$-5x - 10$ . .	-10	-30

When both graphs are plotted on the same axes and using the same scales, we obtain Fig. 74

The graphs intersect at a point whose x value is $-1{\cdot}1$ (approx.). At this point

$$3x^3 = -5x - 10$$

or

$$3x^3 + 5x + 10 = 0$$

$\therefore$ The root of $3x^3 + 5x + 10 = 0$ is $x = -1{\cdot}1$ (approx.)

These methods are both of use in a general way, and by their application (whichever may seem preferable) it is often possible to solve an equation which might be much more difficult to solve algebraically.

9. Graph of $y = ax^3 + bx + c$, where b is negative

Suppose we require the graph of

$$y = 2x^3 - 7x - 3$$

Tabulating values as before, we obtain:

x . .	-2	$-1{\cdot}5$	-1	$-0{\cdot}5$	0	$0{\cdot}5$	1	$1{\cdot}5$	2	$2{\cdot}5$
$2x^3$ . . $-7x$. -3 . .	-16 $+14$ -3	$-6{\cdot}75$ $+10{\cdot}5$ -3	-2 $+7$ -3	$-0{\cdot}25$ $+3{\cdot}5$ -3	0 0 -3	$0{\cdot}25$ $-3{\cdot}5$ -3	2 -7 -3	$6{\cdot}75$ $-10{\cdot}5$ -3	16 -14 -3	$31{\cdot}25$ $-17{\cdot}5$ -3
y . .	-5	$0{\cdot}75$	2	$0{\cdot}25$	-3	$-6{\cdot}25$	-8	$-6{\cdot}75$	-1	$10{\cdot}75$

Here we note that

(a) y changes sign in three places:

(i) between $x = -2$ and $x = -1{\cdot}5$, the change in y being from negative to positive;

(ii) between $x = -1$ and $x = 0$, y changing from positive to negative;

(iii) between $x = 2$ and $x = 2{\cdot}5$, y changing from negative to positive.

(*b*) These results suggest two turning points on the curve:

(i) between $x = -1{\cdot}5$ and $x = -0{\cdot}5$, which will be a *maximum*;

(ii) between $x = 0{\cdot}5$ and $x = 1{\cdot}5$, a *minimum* point.

Other points might be plotted to confirm this and obtain the turning points more exactly.

On drawing the curve (Fig. 75) we find it crosses the axis of x at the points where $x = -1{\cdot}6$, $-0{\cdot}45$ and $2{\cdot}15$,

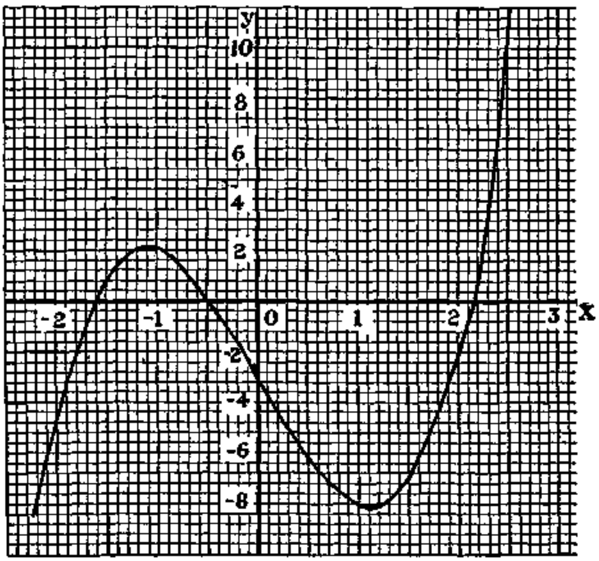

FIG. 75.—Graph of $y = 2x^3 - 7x - 3$.

reaching a maximum value of 2·1 when $x = -1·1$ and a minimum value of $-8·1$ when $x = 1·1$. We also note that $x = -1·6, -0·45$ and $2·15$ are the roots of the equation $2x^3 - 7x - 3 = 0$.

10. Alternative Method of solving $2x^3 - 7x - 3 = 0$

Writing $2x^3 = 7x + 3$, we may use the method of the previous example; by drawing the graphs of $y = 2x^3$ and

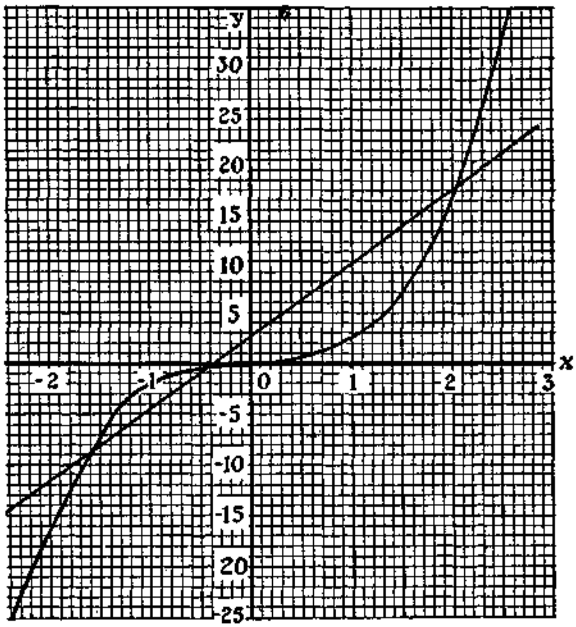

FIG. 76.—Graph showing points of intersection of $y = 2x^3$ and $y = 7x + 3$.

$y = 7x + 3$ we can find their points of intersection, whose x values will be the above roots (Fig. 76).

We find that the straight line cuts the cubic curve in three points, whose x values are $-1\cdot6$, $-0\cdot45$ and $2\cdot15$. These agree with the values previously found for the roots of the equation.

In this particular case we find three real roots of the equation, but in other positions of the straight line—*i.e.*, for other values of b and c—it might cut the cubic curve in *one* point only: there would then be only *one real* root of the equation: the two others would be imaginary.

Thus a cubic equation of the type $ax^3 + bx + c = 0$ may, in general, have three real roots or one real and two imaginary roots.

11. The Graph of $y = x(x - a)(x - b)$

When the function is given in this form, we can get a clear idea of the shape of the curve by inspection.

Suppose we require the graph of

$$y = 4x(x - 1)(x - 2)$$

By inspection we note

(1) **When $x = 0$, 1 or 2, $y = 0$.**

$\therefore$ the curve cuts the x axis at these points.

(2) When **x is negative,** both $(x - 1)$ and $(x - 2)$ are negative. $\therefore$ **y is negative** for all negative values of x.

(3) When **x lies between 0 and 1,** x is positive, while $(x - 1)$ and $(x - 2)$ are both negative. $\therefore$ **y is positive.**

(4) When **x lies between 1 and 2,** x is positive, $(x - 1)$ is positive, but $(x - 2)$ is negative. $\therefore$ **y is negative.**

(5) When **x is greater than 2,** x, $x - 1$ and $x - 2$ are all positive. $\therefore$ **y is positive.**

(6) Since y is positive for values of x between 0 and 1 and zero at these points, the curve (if continuous) must

pass through a turning-point which is clearly a **maximum point** between these two points.

(7) Similarly, the graph will have a turning-point between $x = 1$ and $x = 2$, and since y is negative between these points, the turning-point will be a **minimum point.**

(8) As x approaches infinity (either positive or negative) y approaches infinity (positive or negative, respectively).

It should be noted that the coefficient 4 does not affect the above arguments; it merely multiplies the y values.

We will now draw the graph of

$$y = 4x(x - 1)(x - 2)$$

in the usual way.

x . . .	$-0{\cdot}25$	0	$0{\cdot}25$	$0{\cdot}5$	$0{\cdot}75$	1	$1{\cdot}25$	$1{\cdot}5$	$1{\cdot}75$	2	$2{\cdot}25$
$y=4x(x-1)(x-2)$	$-2{\cdot}81$	0	$1{\cdot}31$	$1{\cdot}5$	$0{\cdot}94$	0	$-0{\cdot}94$	$-1{\cdot}5$	$-1{\cdot}31$	0	$2{\cdot}81$

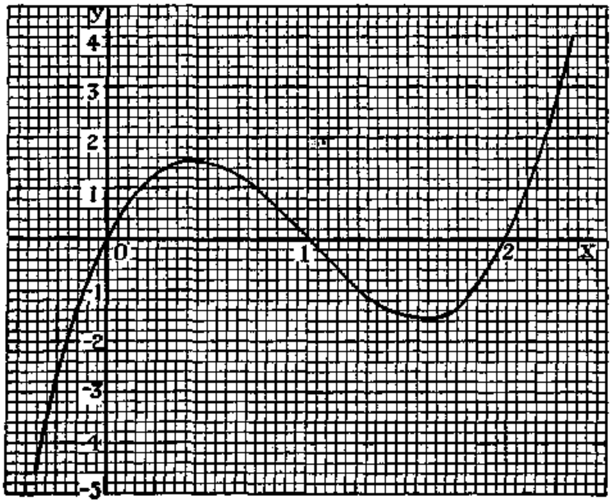

FIG. 77.—Graph of $y = 4x(x - 1)(x - 2)$.

The graph obtained (Fig. 77) verifies the above general observations. The equation

$$4x(x - 1)(x - 2) = 0$$

is shown to have three real roots—namely, $x = 0$, 1 and 2.

The maximum and minimum values of the expression

$$4x(x - 1)(x - 2)$$

are given by the values of y at the turning-points; they are

(*a*) Maximum value, 1·55, when $x = 0{\cdot}45$.
(*b*) Minimum value, $-$ 1·55, when $x = 1{\cdot}55$.

12. The Graph of $y = ax^3 + bx^2 + cx + d$

Lastly we will consider the graph of an expression of the type in which the variable x occurs in the first, second and third degrees.

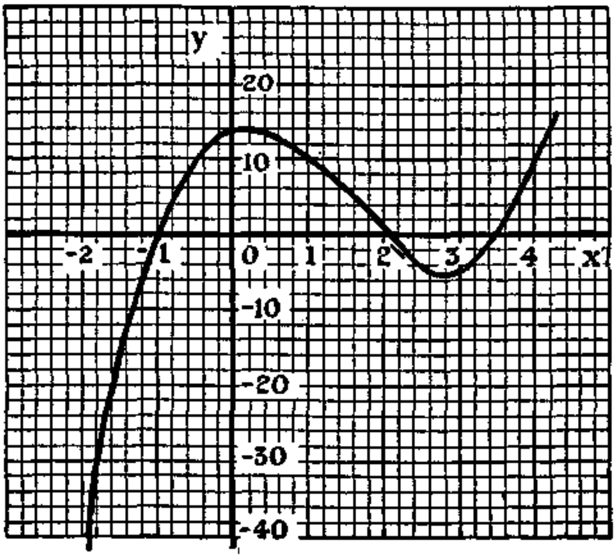

FIG. 78.—Graph of $y = 2x^3 - 9x^2 + 3x + 14$.

Let $a = 2$, $b = -9$, $c = 3$, and $d = 14$.

The graph of

$$y = 2x^3 - 9x^2 + 3x + 14$$

is shown in Fig. 78. It is a curve with two turning-points; it cuts the x axis at $x = -1$, 2 and 3·5.

Hence the roots of the equation $2x^3 - 9x^2 + 3x + 14 = 0$, are $x = -1, 2$ *or* $3·5$.

Thus we see how a cubic equation may be solved or a cubic expression factorised generally, by means of its graph.

13. Alternative Method of solving $2x^3 - 9x^2 + 3x + 14 = 0$

Writing

$$2x^3 = 9x^2 - 3x - 14$$

we can draw the graphs of

$$(a)\ y = 2x^3, \qquad (b)\ y = 9x^2 - 3x - 14$$

and find their points of intersection as in previous examples.

x	-2	-1	0	1	2	3	4
$2x^3$	-16	-2	0	2	16	54	128
$9x^2$	36	9	0	9	36	81	144
$-3x$	6	3	0	-3	-6	-9	-12
-14	-14	-14	-14	-14	-14	-14	-14
$9x^2-3x-14$	28	-2	-14	-8	16	58	118

Fig. 79 shows the points of intersection, which are three in number, when $x = -1$, 2 and 3·5. These are therefore the roots of the equation $2x^3 - 9x^2 + 3x + 14 = 0$ (as found in the first method).

In this case we note that the graphs are the cubic curve (as before) and a parabola; if the parabola had its vertex

(turning-point) *above* 0 on the y axis, instead of *below*, the parabola would cut the cubic curve *once* only: thus we

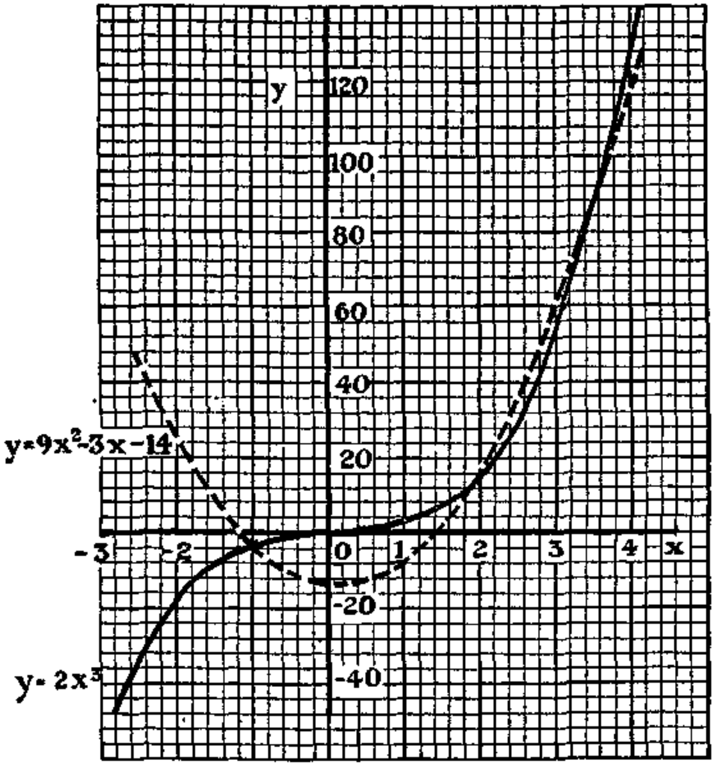

FIG. 79.—Graph showing intersections of $y = 2x^3$ and $y = 9x^2 - 3x - 14$.

should find *one* real root instead of three as above (two roots would be imaginary).

This conclusion could also be verified by reference to Fig. 80, for if the graph (A) with a different value of d were

-11539-

transferred vertically upwards to position (C), the minimum point would lie above the x axis, which would then be cut by the curve in *one* point only, instead of the three previously obtained. (See Fig. 78.)

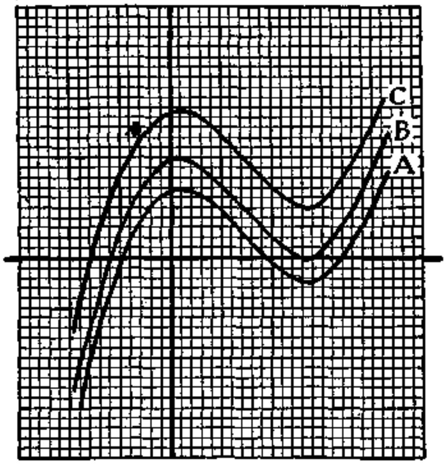

FIG. 80.

As a particular case, we might imagine the minimum point lying *on* the x axis, in which case the two points of intersection of the curve and the x axis would coincide (B); there would thus be three roots, two of which are coincident in value.

14. Other Types of Graphs Commonly Used

Two other types of graphs which the student will often meet are:

$$(a)\ y = ax^n. \qquad (b)\ y = ae^{bx}.$$

Example (a) *Draw the graph of* $y = 3{\cdot}5x^{2{\cdot}8}$.

A graph of this type needs a rather more difficult cal-

culation than the cubic types, as the values of $x^{2\cdot8}$ can be plotted only by using logarithms.

Taking logs. of both sides,

$$\log y = \log 3{\cdot}5 + 2{\cdot}8 \log x$$

Our table of values could be set out as follows:

x	0	0·5	1	1·5	2	2·5	3
$\log x$ . .	$-\infty$	$\bar{1}{\cdot}6990 = -0{\cdot}3010$	0	0·1761	0·3010	0·3979	0·4771
$2{\cdot}8 \log x$ . .	$-\infty$	$-0{\cdot}8428$	0	0·4931	0·8428	1·1141	1·3359
$\log 3{\cdot}5$ . .	0·5441	0·5441	0·5441	0·5441	0·5441	0·5441	0·5441
$\log y$ . .	$-\infty$	$\bar{1}{\cdot}7013$	0·5441	1·0372	1·3869	1·6582	1·8800
y	0	0·5027	3·500	10·89	24·37	45·52	75·86

The graph is shown in Fig. 81.

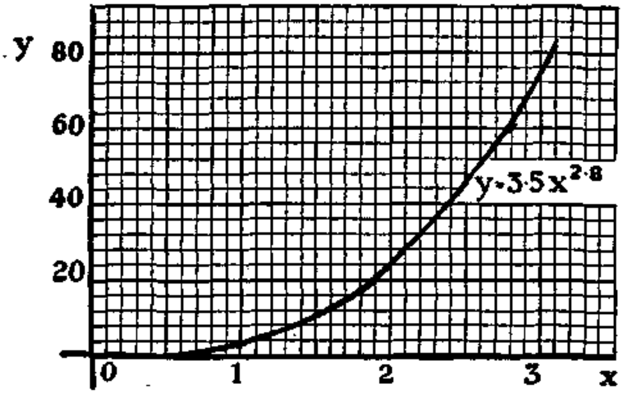

FIG. 81.—Graph of $y = 3{\cdot}5x^{2\cdot8}$.

Example (b) *Draw the Graph of* $y = 3e^{2x}$.

The same method of calculation is used in this case.

Taking logs. of both sides,

$$\log y = \log 3 + 2x \log e$$

Tabulating as before and noting that $\log_{10} e = 0{\cdot}4343$.

x	0	0·25	0·5	0·75	1	1·25	1·5	1·75	2
$2x \log e$ $\log 3$ $\log y$	0 0·4771 0·4771	0·2172 0·4771 0·6943	0·4343 0·4771 0·9114	0·6513 0·4771 1·1286	0·8686 0·4771 1·3457	1·0858 0·4771 1·5629	1·3029 0·4771 1·7800	1·5201 0·4771 1·9972	1·7372 0·4771 2·2143
y	3	4·946	8·155	13·45	22·17	36·56	60·26	99·36	163·8

The graph is shown in Fig. 82.

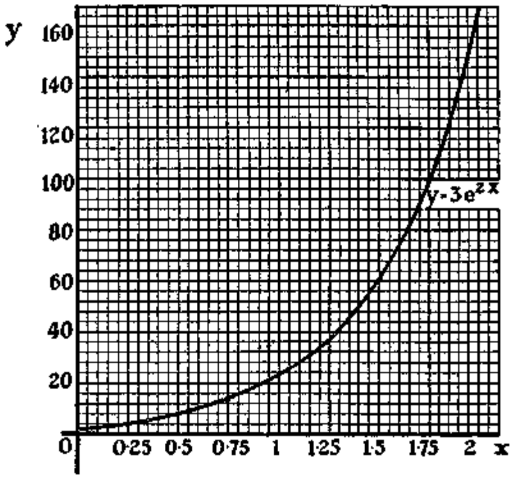

FIG. 82.—Graph of $y = 3e^{2x}$.

EXERCISE 16

1. Draw the graph of $y = 6x^2 + 17x - 45$ from $x = -5$ to $x = 4$. Use it to solve the equations:

(a) $6x^2 + 17x - 45 = 0$.
(b) $6x^2 + 17x = 25$.

What is the minimum value of $6x^2 + 17x - 45$?

2. Draw the graphs of $6x^2$ and $(45 - 17x)$ using the same scales and axes. What are the values of x where the graphs intersect? How do these values compare with the roots of $6x^2 + 17x - 45 = 0$ in Question 1 above?

3. Draw the graph of $y = 7{\cdot}2x - 1{\cdot}5x^2$ from $x = 0$ to $x = 4$. Find the maximum value of $7{\cdot}2x - 1{\cdot}5x^2$ and give the corresponding value of x.

4. In a certain steam-engine the expansion ratio, r, and the number of lb., N, of steam used per I.H.P. per hour, were found to be as follows:

r . . .	3·8	4·2	4·5	4·8	5·0	5·6	5·9	6·3
N . . .	20·9	20·6	20·42	20·3	20·21	20·17	20·27	20·5

Find for what value of r the quantity of steam used is a minimum. What is the minimum value of N then used?

5. Given that $Y = 3M^2 + \frac{75}{M}$, find a value of M which makes Y a minimum. What is the minimum value of Y? (Draw the graph from $M = 0$ to $M = 5$.)

6. Given,

$$P = \frac{36R}{(R + 0{\cdot}32)^2}$$

plot the graph from $R = 0{\cdot}1$ to $R = 0{\cdot}6$, and find what value of R gives a maximum value for P.

7. Solve, graphically, the equation

$$(x)(x - 3)(2x + 8) = 0,$$

and find the maximum and minimum values of the function $(x)(x - 3)(2x + 8)$.

8. Draw the graph of $y = 1{\cdot}7x^{1\cdot 6}$, and hence solve the equation $1{\cdot}7x^{1\cdot 6} = 8{\cdot}8$. (Draw the graph from $x = 0$ to $x = 4$.)

9. Draw the graph of $2{\cdot}2x^{1\cdot 2}$ from $x = 0$ to $x = 5$. Use it to solve $2{\cdot}2x^{1\cdot 2} - 10 = 0$.

10. Graph the functions $\frac{1}{x}$ and $[0{\cdot}4(x + 1)(3 - x)]$ for values of x from -2 to $+4$, using the same scales and reference axes for both. By means of the graphs, estimate to within $\pm$ 0·05. the roots of the equation

$$2x(x + 1)(3 - x) = 5. \quad \text{(N.C.T.E.C., 1935.)}$$

11. The volume in cubic inches of a gas cylinder is given by :

$$V = \frac{11x}{3}\left(21 - \frac{4x^2}{7}\right), \text{ where } x \text{ is in inches.}$$

Taking values of x from 1 to 4, calculate V and tabulate. Show on a diagram how V varies, as x varies from 1 to 4. From your diagram read off : (i) The value of x that gives a maximum volume. (ii) The maximum volume.

(U.E.I., 1935.)

12. The following table gives related values of percentage efficiencies and speed ratios for two turbines A and B.

Speed ratios . . .	0·2	0·4	0·6	0·8	0·9	1·0	1·2	1·4	1·6	1·8
Efficiency of A . .	60	90	90	58	34	0				
Efficiency of B . .	38	64	68	76	80	82	87	90	92	94

Plot, on a speed ratio base, curves connecting efficiency and speed ratio for both turbines.

From your curves determine:

(*a*) the maximum efficiency of A and the corresponding speed ratio;

(*b*) the speed ratio when the efficiencies of A and B are equal;

(*c*) the speed ratio at which the efficiency of B is equal to the maximum efficiency of A.

(U.L.C.I., 1936.)

13. Plot the curve of the equation

$$y = 2x^3 - 5x^2 + 2$$

between the values $x = -2$ and $x = 3$, and from your curve find the roots of the equation

$$2x^3 - 5x^2 + 2 = 0$$

which lie within the range given. (U.L.C.I., 1928.)

14. Find, to two significant figures, the values of x between $+3$ and -3 which satisfy the equation

$$x^3 - 6x + 3 = 0 \qquad \text{(U.L.C.I., 1935.)}$$

15. Graph the function

$$0{\cdot}1(x - 1)(2x + 3)(2x - 7)$$

for all values of x from -2 to $+4$, using 1 in. as the unit along each axis. By means of the graph, estimate to within $\pm 0{\cdot}05$ the roots of the equation

$$(x - 1)(2x + 3)(2x - 7) = 5$$

(N.C.T.E.C., 1933.)

16. Graph the function

$$\tfrac{1}{5}(x - 4)^2(x + 2)$$

for values of x from -2 to 6. By means of the graph, solve each of the equations

(1) $(x - 4)^2(x + 2) = 10$

(2) $(x - 4)^2(x + 2) = 17$. (N.C.T.E.C., 1931.)

17. The formula

$$T = 1700 + \frac{160{,}000}{x^2} - 0{\cdot}6x^2$$

was employed to calculate the hoop tensile stress in a rotating disc. $T =$ the tensile strength in lb., $x =$ the radius in ins. Determine the values of T for the following values of x, viz. :

30, 40, 50 and 60

Express graphically the relation between T and x. For what value of x is $T = 0$? (U.L.C.I., 1927.)

18. Show by means of a diagram how the function $10x - \frac{x^3}{4}$ varies as x varies from 0 to 7.

From the diagram read off the value of x for which the function is a maximum. (U.E.I.)

CHAPTER 12

DETERMINATION OF LAWS

1. Having considered the methods of drawing various graphs from given equations (or given functions of x) we will now consider the converse type of problem.

In this case, we are given the graph, or more usually sets of corresponding values of the two variables, and we are required to find the relation, or equation, connecting them. Usually the sets of corresponding values are the result of experiment, and it may be very useful to find a law connecting them. For example, the student may have measured the different values of an electric current (C) in a circuit containing a constant voltage, when the resistance (R) in the circuit was repeatedly altered: by tabulating the corresponding values of current (C) and resistance (R), he is able to find how they are related to each other, or as the problem is usually stated, "to determine the law" connecting C and R. Once the law has been determined, it serves as a useful basis for further calculations.

2. The Linear Law, $y = ax + b$

This law has already been dealt with in the First-Year Course, where it was shown that all straight lines are represented generally by the equation, $y = ax + b$, the gradient of the line and its actual position depending on the values of a and b, respectively. Thus $y = 3x + 2$, $y = 5 - 4x$, $y = \frac{x}{3} - 6$ are straight lines in which the gradients are $3, -4, \frac{1}{3}$ and the intercepts on the y axis are $2, 5, -6$ respectively. If, then, we plot a given set of

corresponding values of two variables (say x and y) and we obtain a straight line, we know that x and y (or other variables taking their places) are connected by a law of the above type and to determine the law is merely to find the particular values of a and b in the equation $y = ax + b$.

Example

Given the following values of x and y, find the law connecting them.

x . . .	-3	-2	-1	0	2	3	4
y . . .	-17	-13	-9	-5	3	7	11

On plotting these values in the usual way, a straight line is obtained (Fig. 83).

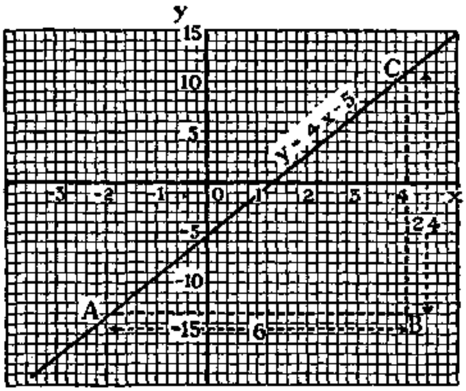

FIG. 83.

Therefore, the equation of the line (or the law connecting x and y) is of the form $y = ax + b$.

The values of a and b can now be found in two ways:

(*a*) By drawing two lines such as AB, BC we find that

the line has risen 24 units (as measured on the y scale) in a horizontal distance of 6 units (measured on the x scale).

$\therefore$ The gradient of the line $= a = \frac{24}{6} = 4$.

Since the line crosses the y axis at -5,

$$b = -5$$

$\therefore$ The equation of the line is

$$y = 4x - 5$$

(*b*) Alternatively, by choosing values from the given table such as $x = -2$ when $y = -13$ and $x = 3$ when $y = 7$, we can substitute in the equation $y = ax + b$ and so obtain the simultaneous equation:

$$7 = 3a + b \quad . \quad . \quad . \quad \text{(A)}$$

$$-13 = -2a + b$$

$$\therefore \quad 20 = 5a$$

$$\therefore \quad a = 4$$

Substituting $a = 4$ in the equation (A)

$$7 = 12 + b$$

$$\therefore \quad b = -5$$

$\therefore$ The equation of the line (or the law connecting x and y) is, as previously found,

$$y = 4x - 5.$$

In actual cases of sets of related quantities found by experiment, the same method is adopted. The values are plotted and if the graph is a straight line (allowance being made for slight divergences owing to experimental error), the values plotted vertically (in the place of y) are connected with those plotted horizontally (in the place of x) by the linear law $y = ax + b$ adapted by putting the variables used in the place of y and x respectively.

If we have plotted a series of values of M vertically and a series of values of N horizontally and the result is a straight line, we know that the law connecting M and N is of the form $M = aN + b$.

3. Laws Other than Linear

Only occasionally do the values give a straight line; generally the graph is curved. A reference to the graphs shown in Figs. 79, 81, 82, in Chapter 11, will convince the student that there is sufficient similarity between the graphs of $y = 2x^3$, $y = 9x^2 - 3x - 14$, $y = 3{\cdot}5x^{2\cdot8}$ and $y = 3e^{2x}$ (or for such portions as can be drawn with the limits of the paper) to make it impossible, by mere inspection, to state definitely the type of law to which a curve, as

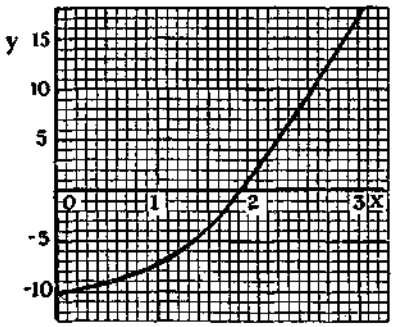

FIG. 84.—Graph of $3x^2 - 10$.

seen in the first quadrant, belongs: for, it must be remembered, usually only that part of the curve which lies in the first quadrant will be obtained by plotting experimental data.

When the graph is a straight line, the form of the law can be definitely stated, but a curved graph may represent any one of a large number of laws, such as $y = ax^2 + b$, $y = ax^3 + bx^2 + cx + d$, $y = ax^n$, $y = ae^{bx}$, etc. To decide the type of law on which the curved graph is based we must adopt a method of "*reducing the curve to a straight line.*" We will illustrate the method by examples.

Consider the graph of $y = 3x^2 - 10$, which we already know is of the parabola type. But if drawn between $x = 0$ and $x = 3$, we should obtain a curve (Fig. 84) which we could not identify merely by inspection *if we did not previously know the type of equation*: it might satisfy equally well one of several other types of laws such as have been dealt with in previous chapters.

But as we already know, in this particular case, that y and x^2 are the forms in which the variables occur in the equation, let us try the effect of plotting the y values against those of x^2 (instead of x).

Tabulating these values we get,

x . . .	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
x^2 . . .	0	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9
$y = 3x^2 - 10$.	-10	$-9\frac{1}{4}$	-7	$-3\frac{1}{4}$	2	$8\frac{3}{4}$	17

When these are plotted, we obtain Fig. 85.

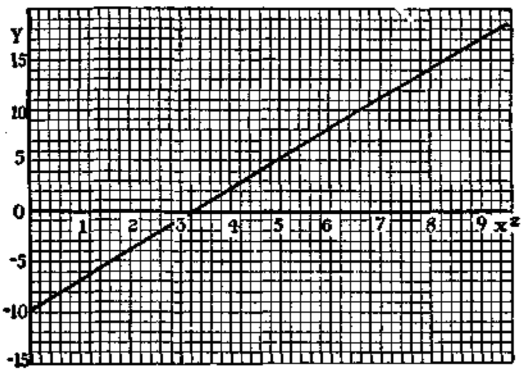

FIG. 85.—Result of plotting y against x^2.

The graph is a straight line

$\therefore$ y is connected with x^2 by the linear form of law

$$\therefore \quad y = ax^2 + b$$

(a result which we already knew, since $a = 3$ and $b = -10$).

Thus if, in future, we can obtain a straight line by plotting some modified forms of y and x (*i.e.*, some other functions of y and x) instead of the y and x values given, we shall be able to deduce the law.

For example, if by plotting M^3 (say) vertically against N^2 horizontally, we obtain a straight line, we infer that

$$M^3 = aN^2 + b$$

If, in general, we plot any particular function of a variable θ vertically against a function of another variable ϕ horizontally, and so obtain a straight line, we shall know that the two functions are connected by the linear law, thus:

$$\textit{the function of } \theta = a \times (\textit{the function of } \phi) + b.$$

Now the problem resolves itself into finding what functions of the variables must be plotted so as to give the straight line. At this stage the student is generally given a hint as to the possible type of relationship, which enables him to avoid cumbersome and tedious trial methods.

Let us consider the following typical examples.

Examples

(1) Type of Law, $y = ax^2 + b$.

The following values of R *and* V *are possibly connected by a law of the type* $R = aV^2 + b$. *Test if this is so, and find the law* (*i.e., find a and b*).

V . . .	12	16	20	22	24	26	30
R . . .	6·44	7·56	9	9·84	10·76	11·76	14

If these values be plotted as given, a curve will result, which tells us nothing, except that they are NOT connected

by the linear law, $y = ax + b$ or $R = aV + b$—a valueless result in view of the suggestion offered in the problem.

But if R be plotted against V^2, a straight line should result. The values of R and V^2 are therefore tabulated and plotted (Fig. 86):

V^2 .	144	256	400	484	576	676	900
R .	6·44	7·56	9	9·84	10·76	11·76	14

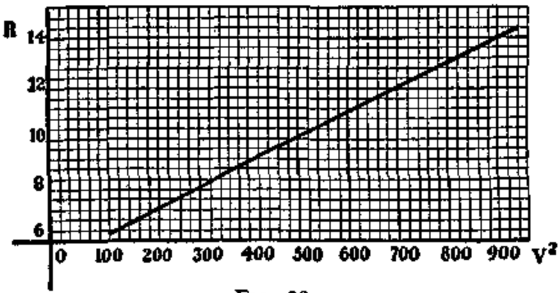

FIG. 86.

The straight line obtained shows that $R = aV^2 + b$.

We can find the values of a and b (and so determine the actual law) by either of the two methods given previously.

Using the simultaneous equation method, we choose two suitable points not too close together, such as (256, 7·56) and (676, 11·76).

$$\therefore \quad 11{\cdot}76 = a \times 676 + b$$
$$7{\cdot}56 = a \times 256 + b \quad . \quad . \quad . \quad (A)$$

or

$$11{\cdot}76 = 676a + b$$
$$7{\cdot}56 = 256a + b$$

By subtraction

$$4{\cdot}20 = 420a$$

$$\therefore \quad a = \frac{4{\cdot}2}{420} = 0{\cdot}01$$

By substitution of $a = 0{\cdot}01$ in equation (A)

$$7{\cdot}56 = (0{\cdot}01 \times 256) + b$$

$$\therefore \quad 7{\cdot}56 = 2{\cdot}56 + b$$

$$\therefore \quad b = 5$$

$\therefore$ *The law is* $R = 0{\cdot}01V^2 + 5$.

Notes.—(*a*) The other method of finding the gradient and the intercept on the y axis could be used as a check on this result.

(*b*) It often happens in practice that the points do not give a perfectly straight line, owing to the fact that experimental readings involve slight errors. In such cases, a line should be drawn intermediate between the points and the simultaneous equation built from values chosen from the line, rather than from the slightly erroneous values given. Of course, the student must first satisfy himself that the points DO represent a straight line with slight divergencies.

(2) Type of Law, $y = \frac{a}{x} + b$.

In measuring the Resistance, R *ohms, of a carbon-filament lamp at various Voltages,* V, *the following results were obtained:*

V (*volts*) . .	*60*	*70*	*80*	*90*	*100*	*120*
R (*ohms*) . .	*70*	*67·2*	*65*	*63·3*	*62*	*60*

Show that the law connecting R *and* V *is of the form*

$$R = \frac{a}{V} + b$$

and then find it.

By comparing

$$y = ax + b \text{ with } R = \frac{a}{V} + b$$

(which, as seen in the First-Year Course, is a hyperbola) we find that:

R occupies the place of y and $\frac{1}{V}$ occupies the place of x in the equation.

We therefore plot R values vertically and values of $\frac{1}{V}$ horizontally.

Tabulating and plotting, we obtain :

V .	60	70	80	90	100	120
$\frac{1}{V}$.	0·0167	0·0143	0·0125	0·0111	0·01	0·0083
R .	70	67·2	65	63·3	62	60

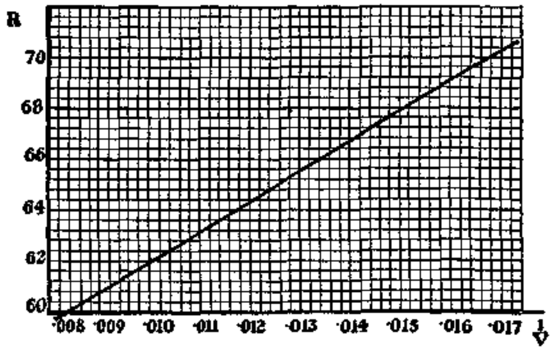

FIG. 87.—$R = \frac{a}{V} + b$.

A straight line is obtained (Fig. 87)

$$\therefore \quad R = a \times \frac{1}{V} + b$$

or $$R = \frac{a}{V} + b \textit{ is the form of the law}$$

To find a and b

Selecting $\frac{1}{V} = 0{\cdot}0143$ when $R = 67{\cdot}2$

and $\frac{1}{V} = 0{\cdot}0083$ when $R = 60$

and substituting in $R = \frac{a}{V} + b$

we obtain :

$$67{\cdot}2 = a \times 0{\cdot}0143 + b$$
$$60 \quad = a \times 0{\cdot}0083 + b$$

or

$$67{\cdot}2 = 0{\cdot}0143a + b$$
$$60 \quad = 0{\cdot}0083a + b \quad . \quad . \quad . \quad \text{(A)}$$
$$\therefore \quad 7{\cdot}2 = 0{\cdot}0060a$$
$$\therefore \quad a = \frac{7{\cdot}2}{0{\cdot}006} = 1200$$

Substituting $a = 1200$ in equation (A)

$$60 = (0{\cdot}0083 \times 1200) + b$$
$$\therefore \quad 60 = 9{\cdot}96 + b$$
$$\therefore \quad b = 60 - 9{\cdot}96 = 50{\cdot}04 = 50 \text{ (approx.)}$$

$\therefore$ The law is

$$R = \frac{1200}{V} + 50$$

(3) Type of Law, $y = ax^n$.

Find the law connecting H and v (it is probably of the form $H = av^n$).

v . .	15	18	20	22	24	25	27
H . .	354	623	863·4	1160	1519	1724	2190

(Note that a and n are constants.)

If the law connecting H and v is of the type suggested, we know that by plotting the given values we should get a curve (of the type shown in Fig. 81), but, as before, we cannot distinguish such a curve from curves depending on other types of laws. Thus, the curve obtained does not indicate the type of law applicable.

To obtain a straight line.

In the given type of equation $H = av^n$ take logs. of both sides,

$$\therefore \quad \log H = \log a + n \log v$$

or

$$\log H = n \log v + \log a$$

By comparison with the linear law $y = ax + b$ we find that the two variables log H and log v are connected in the same way as y and x.

$\log H = \log v \times$ (a constant, n) + (a constant, $\log a$)
corresponding to

$$y = x \times \text{(a constant, } a\text{)} + \text{(a constant, } b\text{)}$$

$\therefore$ If log H is plotted against log v, a straight line should be obtained, if the suggested law holds good.

We therefore tabulate log H and log V and plot in the usual way.

log v	1·1761	1·2553	1·3010	1·3424	1·3802	1·3979	1·4314
log H	2·5490	2·7945	2·9362	3·0645	3·1816	3·2365	3·3404

The graph is a straight line (Fig. 88).

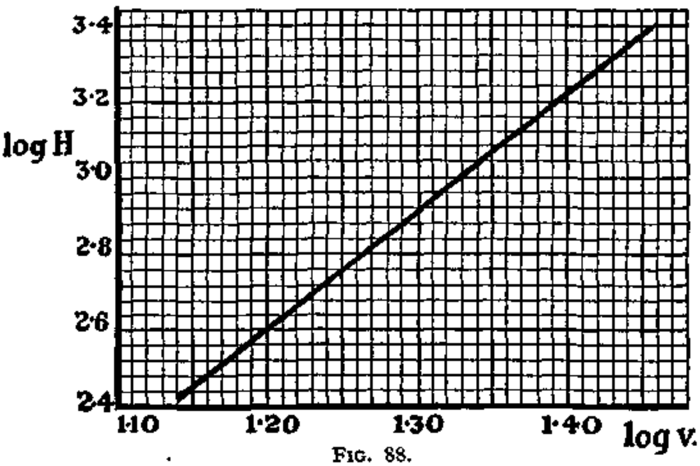

FIG. 88.

$$\therefore \quad \log H = n \log v + \log a$$

where n and a are constants.

$$\therefore \quad H = av^n \textit{ is the type of law}$$

The constants a and n are found in the usual way.

Choosing $\log H = 3{\cdot}2365$ when $\log v = 1{\cdot}3979$

and $\qquad \log H = 2{\cdot}7945$ when $\log v = 1{\cdot}2553$

and substituting in $\log H = n \log v + \log a$.

$$3{\cdot}2365 = n \times 1{\cdot}3979 + \log a$$

$$2{\cdot}7945 = n \times 1{\cdot}2553 + \log a \quad . \quad . \quad \text{(A)}$$

$$\therefore \quad 0{\cdot}4420 = 0{\cdot}1426n$$

$$\therefore \quad n = \frac{0{\cdot}4420}{0{\cdot}1426} = 3{\cdot}1$$

Substituting this value of n in equation (A)

$$2{\cdot}7945 = (3{\cdot}1 \times 1{\cdot}2553) + \log a$$

$$\therefore \quad 2{\cdot}7945 = 3{\cdot}8914 + \log a$$

$$\therefore \quad \log a = 2{\cdot}7945 - 3{\cdot}8914$$

$$\therefore \quad \log a = \bar{2}{\cdot}9031$$

$$\therefore \quad a = 0{\cdot}08$$

$\therefore$ The law connecting H and v is

$$H = 0{\cdot}08v^{3{\cdot}1}$$

(4) Type of Law, $y = ae^{bx}$.

Show that the following values of x and y are connected by a law of the form $y = ae^{bx}$ and find the constants a and b.

x . .	2	2·5	3	3·5	4	4·5	5
y . .	30·26	47·44	74·47	116·7	182·8	287·1	449·8

If $\qquad y = ae^{bx}$

then $\qquad \log y = \log a + bx \log e$

or writing $\qquad b \log e = B$

$$\therefore \quad \log y = \log a + Bx$$

Comparing this with $y = a + bx$ we find that

$$\log y = \log a + Bx$$

is the linear law, where $\log y$ and x are the variables, $\log a$ and B being constants.

$\therefore$ If we plot $\log y$ against x (not $\log x$), we should get a straight line, if the suggested law applies.

Tabulating and plotting as below:

x . .	2	2·5	3	3·5	4	4·5	5
$\log y$ . .	1·4809	1·6762	1·872	2·067	2·262	2·458	2·653

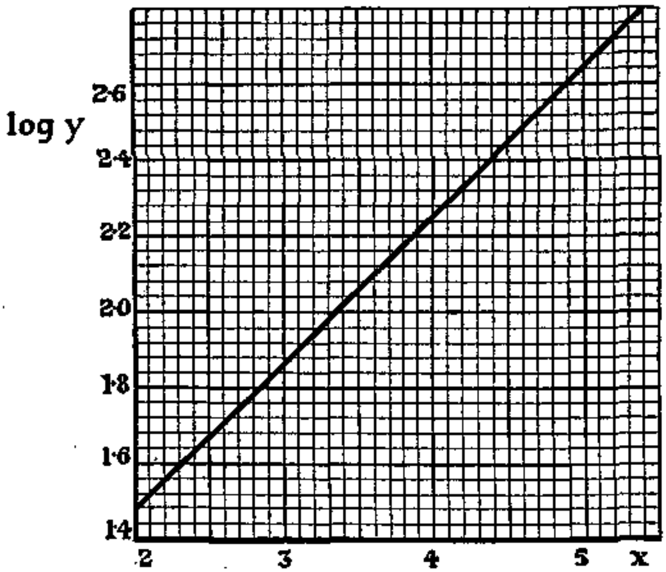

FIG. 89.

The graph is a straight line (Fig. 89)

$\therefore \quad \log y = \log a + Bx$ (where $B = b \log e$)

$\therefore \quad \log y = \log a + bx \log e$

$\therefore \quad y = ae^{bx}$

Substituting selected values of x and $\log y$ in the equation

$$\log y = \log a + Bx$$

$$2{\cdot}4580 = \log a + B \times 4{\cdot}5$$

$$1{\cdot}6762 = \log a + B \times 2{\cdot}5 \quad . \quad . \quad . \quad (A)$$

$$\therefore \quad 0{\cdot}7818 = 2B$$

$$\therefore \quad B = \frac{0{\cdot}7818}{2} = b \log e$$

$$\therefore \quad b = \frac{0{\cdot}7818}{2 \log e} = 0{\cdot}9001 = 0{\cdot}9 \text{ (approx.)}$$

To find a, substitute $B = \frac{0{\cdot}7818}{2}$ in equation (A)

$$\therefore \quad 1{\cdot}6762 = \log a + \frac{0{\cdot}7818}{2} \times 2{\cdot}5$$

$$\therefore \quad 1{\cdot}6762 = \log a + 0{\cdot}9772$$

$$\therefore \quad \log a = 1{\cdot}6762 - 0{\cdot}9772$$

$$\therefore \quad \log a = 0{\cdot}6990$$

$$\therefore \quad a = 5{\cdot}00$$

$\therefore$ The required constants are

$$a = 5, \; b = 0{\cdot}9$$

or the law is

$$y = 5e^{0{\cdot}9x}$$

4. We have examined five types of laws and the methods of determining the actual laws by finding the constants.

(1) $y = ax + b$.
(2) $y = ax^2 + b$ (of which $R = aV^2 + b$ is a special case).
(3) $y = \frac{a}{x} + b$ $\left(\text{of which } R = \frac{a}{V} + b \text{ is a special case}\right)$.
(4) $y = ax^n$ (of which $H = av^n$ is a special case).
(5) $y = ae^{bx}$.

In every case except the first the curve was reduced to a straight line, each type requiring its own particular treatment, but the methods used will suggest ways of dealing with other forms as they arise.

5. Summary

If a suggested type of law be compared with $y = ax + b$, the functions which occupy the places of y and x, respectively, when plotted in those places, should give a straight line.

Conversely, if by trial we obtain a straight line by plotting a function of y against a function of x, then these functions are connected by the linear law.

EXERCISE 17

1. Prove that the following values of x and y are connected by the law $y = ax + b$. Find the constants a and b.

x . .	-4	-2	0	2	4	6	8
y . .	$-19{\cdot}2$	$-14{\cdot}6$	-10	$-5{\cdot}4$	$-0{\cdot}8$	$3{\cdot}8$	$8{\cdot}4$

2. The following are values of L, the latent heat of steam, at corresponding temperatures $\theta°$ C. Find the law connecting L and θ in the form $L = a + b\theta$, and find the latent heat of steam at 100° C.

$\theta°$ C. . . .	70	80	90	110	130	140
L (calories) . .	558	551	545	530	516	509

3. Complete the following table of values, given that y and x are connected by a law of the form $y = ax + b$, where a and b are constants.

x . .	-1	0		$1{\cdot}4$	3	5	7
y . .			0		12		32

(N.C.T.E.C., 1934.)

4. The following table gives the result of an experiment. Plot the load-effort diagram and determine the equation (of the form $E = mW + c$) which most nearly accords with the results.

W (load in lb.) .	7	14	21	28	35	42	49	56
E (effort in lb.) .	3	6·5	9·5	12·5	16	18·75	22	25

(U.L.C.I., 1928.)

5. By experiment, the following relation is found between two quantities R and D.

R	10·5	15·8	21·3	26·6	31·6	36·4	42
D	0	1	2	3	4	5	6

It is believed that this relation can be expressed by an equation of the form $R = mD + c$.

Plot the results and from your graph determine the values of m and c. (U.L.C.I., 1927.)

6. Complete the following table, given that y and x are connected by a law of the form $\frac{y}{a} = \frac{1}{x + b}$.

x	-1			2	3	6
y		4	2·5	2		1

(N.C.T.E.C., 1935.)

7. The following table gives related values of x and y. Determine whether these values are connected by an equation of the form $y = ax^2 + b$, where a and b are constants, and if so, find the values of a and b.

x	4	5	6	7	8	9
y	14·3	18	22·5	28	34·5	41·5

(U.L.C.I., 1936.)

8. The following values of R and V satisfy a law of the form $R = a + bV^2$, where a and b are constants.

V	20	25	30	35	40	45
R	52	58	67	76	88	100

Plot suitable quantities to obtain a straight-line graph and obtain the values of a and b. (U.L.C.I., 1935.)

9. A price-list of spanners contains the following sizes and prices. Find as accurately as possible the law connecting S and P (probably in the form $P = aS^2 + b$).

S (sizes) . . .	$\frac{1}{2}''$	$\frac{3}{4}''$	$1''$	$1\frac{1}{4}''$	$1\frac{1}{2}''$	$1\frac{3}{4}''$	$2''$
P (prices) . .	1/4	1/8	2/3	2/10½	3 8	4/8	5/10

10. In an experiment, the following values of N and D were obtained. There is reason to suspect that they are connected by a formula of the type $N = a + \frac{b}{D}$. Test if this is so, and find a and b.

N	0·322	0·317	0·313	0·311	0·309	0·308
D	0·45	0·6	0·75	0·9	1·05	1·2

11. A certain type of vessel needs horse-power, P, to drive it at a speed, V knots. Find the relation between P and V using the following values.

V	10	12	14	15	17
P	1500	2300	3400	4000	5700

(The relation is believed to be of the form $P = mV^3 + a$.)

12. The law connecting the coefficient of friction, μ, between a belt and pulley, with the velocity, v, of the pulley in ft. per minute is believed to be of the form

$$\mu = a\sqrt{v} + b$$

Test if the following values of μ and v agree with this law and find a and b.

μ	0·210	0·223	0·235	0·245	0·262	0·277
v	400	600	800	1000	1400	1800

13. Air is being compressed adiabatically and the pressure and temperature are measured. The law connecting them is believed to be $T = ap^n$, where T is the absolute temperature (*i.e.*, temperature, t, in degrees Centigrade + 273) and p is the pressure in lb. per sq. in. From the following numbers, obtain the best values of a and n.

p (lb. per. sq. in.) . .	15	25	45	70	90
t °C.	200	280	385	475	537

14. The following values of H and Q are connected by a law of the type $Q = aH^n$. Find a and n.

H	1·2	1·6	2·0	2·2	2·5	3·0
Q . . .	6·087	6·751	7·316	7·571	7·927	8·467

15. Given that $T = ae^{\mu\theta}$, find a and μ.

θ . . .	0·35	0·698	1·047	1·396	1·745
T . . .	17·77	19·74	21·91	24·32	27

16. D is the diameter of shafting required to transmit horse-power H at a certain number of revolutions per minute. The law connecting D and H is $D = aH^b$, where a and b are constants. Find a and b by using the following values. There may be errors of observation.

H . .	15	20	25	30	35	40	45	50
D . . .	2·6	2·9	3·2	3·4	3·65	3·85	4	4·2

17. The following table shows the pressure P at various heights, h, above sea-level. The law connecting P and h is $P = Ae^{kh}$. Find the law.

h (ft.) . . .	2000	4000	5000	7000	8000	9000	10000
P (ins. of mercury) .	27·74	25·66	24·69	22·83	21·96	21·12	20·31

18. Experiments have shown that a suitable speed N (revs. per min.) of a twist drill of diameter d (ins.) operating on mild steel with a proper feed is given in the following:

N . . .	265	125	80	65
d . . .	1	2	3	4

It is thought that these results may be expressed in law form, $N = \dfrac{\text{constant}}{d^{1\cdot125}}$.

Calculate for each value of d the value of $\dfrac{1}{d^{1\cdot125}}$ and tabulate. Now plot to as big a scale as the paper will allow, N vertically and $\dfrac{1}{d^{1\cdot125}}$ horizontally. From your graph read off the average value of the constant. (U.E.I., 1932.)

19. A steamship at the following speeds (v knots) uses the following indicated horse-power P :

v . . .	10	12	14	16	18	20
P . . .	1066	1912	3216	4951	7361	10355

Find if there is a law of the form $P = av^n$, and if so, what are the most probably correct values of a and n? There are experimental errors in the observed values of v and P. (B.E.)

20. The following corresponding values of x and y were measured. There may be errors of observation. Test if there is a probable law $y = a + bx^2$ and, if this is the case, what are the probable values of a and b?

x . .	1·00	1·50	2·00	2·30	2·50	2·70	2·80
y . .	0·77	1·05	1·50	1·77	2·03	2·25	2·42

(B.E.)

21. In some experiments in towing a canal boat the following observations were made, P being the pull in pounds and v the speed of the boat in miles per hour :

v	1·68	2·43	3·18	3·60	4·03
P	76	160	240	320	370

Plot $\log v$ and $\log P$ upon squared paper, and give an approximate formula connecting P and v. (B.E.)

22. The following numbers are authentic; t secs. is the record time of a trotting (in harness) race of m miles :

m .	1	2	3	4	5	10	20	30	50	1[illegible]
t .	119	257	416	598	751	1575	3505	6479	14141	3215[illegible]

It is found that there is approximately a law $t = am^b$, where a and b are constants. Test if this is so, and find the most probable values of a and b. The average speed in a race is $S = \frac{m}{t}$: express S in terms of m. (B.E.)

CHAPTER 13

THE BINOMIAL THEOREM

1. Binomial Expressions

A binomial expression is an algebraic expression consisting of the sum of two terms;

$$a + b,\ \ x - y,\ \ 3m - 2n,\ \ 5x^2 + 3y^2$$

are binomial expressions.

In general, any such expression can be denoted by

$$a + b$$

2. Expansions of $(a + b)$

It is sometimes necessary to find various powers of $(a + b)$ and similar expressions. This can be done by actual multiplication and the student will probably be familiar with the following expansions:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

etc.

The process of finding higher powers in this way soon becomes laborious.

3. The Binomial Theorem

The Binomial Theorem is a general formula for such expansions. Its proof will be given in Vol. III. For the present we merely state the theorem. It is as follows:

$$(a + b)^n = a^n + n\,.\,a^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}b^4 + \ldots$$

A special form of this, of considerable practical importance, is obtained when $a = 1$ and $b = x$.

Then the theorem becomes

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$

Similarly if x be replaced by $-x$

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{1 \cdot 2}x^2 - \frac{n(n-1)(x-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$

This theorem is true under certain conditions for positive, negative, integral and fractional values of n, but a formal proof is beyond the scope of this book.

At this stage we shall use it

(*a*) to expand simple binomial expressions;

(*b*) to determine the value of a particular term in a given expansion without writing the whole expansion;

(*c*) for certain arithmetical operations. The methods are indicated in the following examples.

Examples

(1) *Expand* $(2x + 3)^5$ *by the Binomial Theorem.*

Comparing with

$$(a + b)^n = a^n + n \cdot a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 \dots$$

we note that

$$a = 2x, \quad b = 3, \quad n = 5$$

Substituting,

$$\begin{aligned}(2x + 3)^5 &= (2x)^5 + 5(2x)^4(3) + \frac{5 \cdot 4}{1 \cdot 2}(2x)^3(3)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(2x)^2(3)^3 \\ &\quad + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}(2x)(3)^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(3)^5 \\ &= (32x^5) + (5 \cdot 16x^4 \cdot 3) + (10 \cdot 8x^3 \cdot 9) \\ &\quad + (10 \cdot 4x^2 \cdot 27) + (5 \cdot 2x \cdot 81) + 243 \\ &= \mathit{32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243}\end{aligned}$$

(2) *Expand*

$$\left(x - \frac{1}{2y}\right)^6$$

Here $a = x$, $b = -\frac{1}{2y}$, $n = 6$.

$$\therefore \left(x - \frac{1}{2y}\right)^6 = x^6 + 6x^5\left(-\frac{1}{2y}\right) + \frac{6.5}{1.2}x^4\left(-\frac{1}{2y}\right)^2$$
$$+ \frac{6.5.4}{1.2.3}x^3\left(-\frac{1}{2y}\right) + \frac{6.5.4.3}{1.2.3.4}x^2\left(-\frac{1}{2y}\right)$$
$$+ \frac{6.5.4.3.2}{1.2.3.4.5}x\left(-\frac{1}{2y}\right)^5 + \frac{6.5.4.3.2.1}{1.2.3.4.5.6}\left(-\frac{1}{2y}\right)^6$$
$$= x^6 - \frac{3x^5}{y} + \frac{15x^4}{4y^2} - \frac{5x^3}{2y^3} + \frac{15x^2}{16y^4} - \frac{3x}{16y^5} + \frac{1}{64y^6}$$

(3) *Write down the first four terms in the expansion of*

$$(x + 2)^{-3}$$

Here $a = x$, $b = 2$, $n = -3$.

$$\therefore (x + 2)^{-3} = x^{-3} + (-3)x^{-4}.2. + \frac{(-3)(-4)}{1.2}x^{-5}.2^2$$
$$+ \frac{(-3)(-4)(-5)}{1.2.3}x^{-6}.2^3 \dots$$
$$= x^{-3} - 6x^{-4} + 24x^{-5} - 80x^{-6} \dots$$
$$= \frac{1}{x^3} - \frac{6}{x^4} + \frac{24}{x^5} - \frac{80}{x^6} \dots$$

Note on Factorial Notation:

Such an expression as $1.2.3.4$ is sometimes written $\underline{|4}$ and is read "factorial 4."

Thus $\underline{|8}$ would mean $1.2.3 \dots 7.8$.
and $\underline{|r}$ would mean $1.2.3.4 \dots r$.

4. Determination of a Particular Term

To determine a particular term in a given expansion without writing the whole series, we use without proof the formula:

$(r + 1)^{\text{th}}$ term of $(a + b)^n$

$$= \frac{(n)(n-1)(n-2) \dots (n-r+1)}{\underline{|r}} a^{n-r} b^r$$

Examples

(1) *Find the ninth term of* $(2x + 1)^{12}$.

Here $a = 2x$, $b = 1$, $n = 12$, $r + 1 = 9$. $\therefore \quad r = 8$.

$\therefore$ Ninth term of $(2x + 1)^{12}$

$$= \frac{12 \,.\, 11 \,.\, 10 \,.\,.\,.\, (12 - 8 + 1)}{\underline{|8}}(2x)^{12-8} \,.\, 1^8$$

$$= \frac{12 \,.\, 11 \,.\, 10 \,.\, 9 \,.\, \cancel{8} \,.\, \cancel{7} \,.\, \cancel{6} \,.\, \cancel{5}}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, \cancel{5} \,.\, \cancel{6} \,.\, \cancel{7} \,.\, \cancel{8}}(2x)^4 \,.\, 1$$

$$= 495 \times 16x^4$$

$$= \mathit{7920}x^4$$

(2) *Find the seventh term of*

$$\left(3m - \frac{2}{n}\right)^{10}$$

Here $a = 3m$, $b = -\dfrac{2}{n}$, $n = 10$, $r + 1 = 7 \quad \therefore \quad r = 6$.

Substituting

$$\text{seventh term} = \frac{10 \,.\, 9 \,.\, 8 \,.\,.\,.\,.\, (10 - 6 + 1)}{\underline{|6}}(3m)^{10-6}\left(-\frac{2}{n}\right)^6$$

$$= \frac{10 \,.\, 9 \,.\, 8 \,.\, 7 \,.\, \cancel{6} \,.\, \cancel{5}}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, \cancel{5} \,.\, \cancel{6}}(3m)^4\left(-\frac{2}{n}\right)^6$$

$$= 210 \times 81m^4 \times \frac{64}{n^6}$$

$$= \frac{\mathit{1088640}m^4}{n^6}$$

(3) *Find the value of* $(1{\cdot}0005)^4$ *to four decimal places.*

$$(1 + 0{\cdot}0005)^4 = 1^4 + 4 \,.\, 1^3 \,.\, (0{\cdot}0005) + \frac{4 \,.\, 3}{1 \,.\, 2} \,.\, 1^2(0{\cdot}0005)^2$$

$$= 1 + (4 \times 0{\cdot}0005) + (6 \times 1 \times 0{\cdot}00000025)$$

$$= 1 + 0{\cdot}0020 + \;.\;.\;.$$

$= \mathit{1{\cdot}0020}$ (to four decimal places)

EXERCISE 18

Write down the expansions in Nos. 1–10.

1. $(x + 3)^5$.
2. $(3x + 2y)^4$.
3. $\left(x + \frac{2}{x}\right)^6$.
4. $(1 - xy)^7$.
5. $(a^2 - 2c)^4$.
6. $(2x - 3y)^3$.
7. $\left(1 - \frac{1}{x}\right)^{10}$.
8. $(\frac{1}{2} + a)^8$.
9. $(x + \Delta x)^{-2}$ to 3 terms.
10. $\frac{1}{(m + n)^4}$ to three terms [*i.e.* $(m + n)^{-4}$].

Evaluate, by using the Binomial Theorem :

11. $(1{\cdot}002)^6$ to three decimal places.
12. $\frac{1}{(0{\cdot}999)^3}$ to three decimal places.
13. $(1{\cdot}000012)^3$ to six decimal places.

Find the following terms :

14. Fifth term of $\left(3a - \frac{b}{2}\right)^7$.
15. Fourth term of $(2x - 3y)^7$.
16. Fifth term of $\left(2a - \frac{b}{3}\right)^8$.
17. Tenth term of $(1 - 2x)^{12}$.
18. Tenth term of $\left(3x + \frac{1}{x}\right)^{12}$.
19. Sixth term of $\left(\frac{1}{3} + \frac{2}{y}\right)^9$.
20. Ninth term of $\left(3x - \frac{b}{3}\right)^{12}$.
21. Obtain the first four terms of the expansion of $(x - 12)^{12}$. (U.L.C.I., 1936.)
22. Expand $(1 + x)^5$ using the Binomial Theorem. (U.C.L.I., 1935.)

CHAPTER 14

FUNCTIONS—RATE OF INCREASE—DIFFERENTIATION

1. Functions—Meaning

The term "function" has been used occasionally in previous chapters, and now it becomes necessary to examine its meaning more closely.

Simple cases of functions may be quoted:

(1) The cost, C, of a number of books at a given price per book, will depend on the number, n.

$\therefore$ C is said to be a function of n.

(2) The area, A, of a square depends on the length of its side, l.

$\therefore$ A is a function of l.

(3) The distance, s, traversed by a falling body depends on the time, t, during which it falls.

$\therefore$ s is a function of t.

(4) If $y = 3x^2 + 2x + 5$, the value of y will depend on the value of x.

$\therefore$ y is a function of x.

(5) The sine of an angle depends upon the angle.

$\therefore$ $\sin \theta$ is a function of θ.

2. Dependent and Independent Variables

It will be noted that in each of the above examples, the value of one of the terms depends on the value chosen for the other; the latter is known as the "independent" variable, the former the "dependent" variable.

Thus in

(1) n is the independent variable, and C the dependent.

(2) l is the independent variable, and A the dependent.
(3) t is the independent variable, and s the dependent.
(4) x is the independent variable, and y the dependent.
(5) θ is the independent variable, and $\sin\theta$ the dependent.

In each case the dependent variable is said to be a *function* of the independent variable.

We can thus arrive at a

3. Definition of a Function

When two quantities are so related that corresponding to any value of the first quantity there is a definite value of the second, then the second quantity is called a function of the first.

4. The Functional Notation

The symbol $f(x)$ is often used to denote "a function of x."

Thus if $\qquad y = 3x^2 + 2x + 5$
we may write $\qquad f(x) = 3x^2 + 2x + 5$

implying that the function of x under consideration is $3x^2 + 2x + 5$.

Other symbols similarly used are $\phi(x)$, $F(x)$, etc.

Thus $\qquad y = 5x + 7$
may be written $\qquad \phi(x) = 5x + 7$
Also $\qquad y = 2\sin x$
may be written $\qquad F(x) = 2\sin x$

During any investigation, it is usual to use the same functional symbol whenever the same relation exists between the function and its variable; thus the same symbol indicates that the same operation (or series of operations) must be performed on the new value of the independent variable to obtain the value of the function.

Thus if, as above,

$$f(x) = 3x^2 + 2x + 5$$

and a be substituted for x we may write

$$f(a) = 3a^2 + 2a + 5$$

Similarly

$$f(m+n) = 3(m+n)^2 + 2(m+n) + 5$$
$$f(b+3) = 3(b+3)^2 + 2(b+3) + 5$$
$$= 3b^2 + 20b + 38$$
$$f(2) = 3(2)^2 + 2(2) + 5$$
$$= 21$$
$$(f0) = 3(0)^2 + 2(0) + 5$$
$$= 5$$

If
$$\phi(x) = 5x + 7$$
$$\phi(c) = 5c + 7$$
$$\phi(0) = 5(0) + 7 = 7$$
$$\phi(-1) = 5(-1) + 7 = 2$$

If
$$F(x) = 2 \sin x$$
$$F(90°) = 2 \sin 90° = 2$$
$$F(0°) = 2 \sin 0° = 0$$
$$F(\pi) = 2 \sin \pi = 0$$

5. Notation for Increases in Functions

A special notation is employed to denote increases in variable quantities.

Thus in considering the distance s passed over by a moving body in time t (s being a function of t), an increase in time is denoted by Δt; the corresponding increase in distance is denoted by Δs.

Generally, if y is a function of x, Δx is taken to represent an increase or increment in x and Δy the corresponding increase or increment in y.

NOTE.—It should be noted that the increment may be positive or negative.

Δ is the Greek letter corresponding to our "D" and Δx does not stand for the product of Δ and x, but is merely a short way of writing "an increment in x." Similarly Δy denotes "an increment in y." Sometimes the symbols δx and δy are similarly used, δ being the small Greek "d."

Using this notation, we may say,

If two variable quantities x and y are so related that y is a function of x, then if x receives an increment Δx, y will receive a corresponding increment Δy.

Thus it follows that if in an expression which expresses y as a function of x, an increment Δx is given to x, then x must be replaced by $(x + \Delta x)$ throughout, and y by $(y + \Delta y)$.

Thus if $y = x^2 + 5x - 2$ and x receives an increment Δx, then y will receive an increment Δy, and the equation becomes $y + \Delta y = (x + \Delta x)^2 + 5(x + \Delta x) - 2$.

Similarly, if the distance s passed over by a moving body in time t be expressed by $s = 16t + 4t^2$ and t and s receive increments Δt and Δs respectively, then

$$s + \Delta s = 16(t + \Delta t) + 4(t + \Delta t)^2$$

6. Gradient and Slope

Consider any two points P and Q on a straight line (Fig. 90).

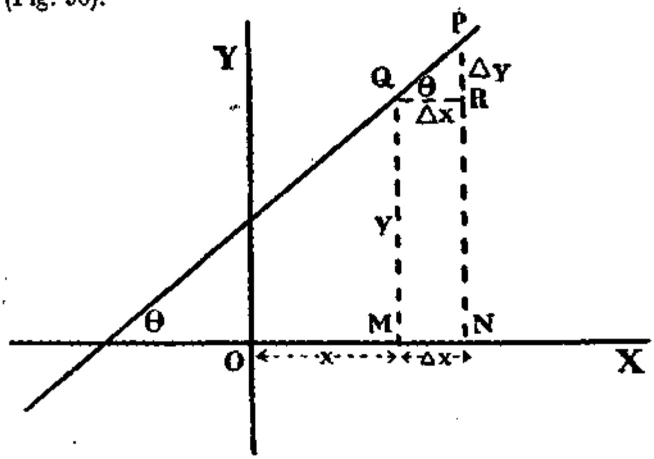

Fig. 90.

Then the ratio $\frac{PR}{QR}$ is constant for all positions of P and Q.

This ratio is called *the gradient of the line.*

If θ be the angle made with the x axis,

then $$\angle PQR = \theta$$

and $$\frac{PR}{QR} = \tan \theta$$

Thus the gradient of a straight line is the tangent of the angle made with the x axis in a positive direction.

The angle θ is called the slope of the line.

7. Constant Gradient of a Straight Line

Since the ratio $\frac{PR}{QR}$ is constant for all positions of P and Q, *the gradient of a straight line is constant.*

If the co-ordinates of Q are denoted by (x, y) let MN(= QR) be represented by Δx (the increase in x) and PR (the increase in y) by Δy, then

the gradient of the line is $\frac{\Delta y}{\Delta x}$ which is equal to $\tan \theta$, where θ is the angle made by the straight line with the x-axis.

8. Gradient of a Curve

To find the gradient of a curve is a more difficult problem than in the case of a straight line, since the curve continuously changes its direction.

In Fig. 91, consider the portion PQ of the curve, cut off by the chord PQ.

Let the co-ordinates of the point P be (x, y).

Then the co-ordinates of Q can be expressed by

$$(x + \Delta x, y + \Delta y)$$

as above.

Then $$\frac{\Delta y}{\Delta x} = \tan \theta,$$

where θ is the angle made by the chord with the x axis.

But $\frac{\Delta y}{\Delta x}$ is the gradient of the chord PQ.

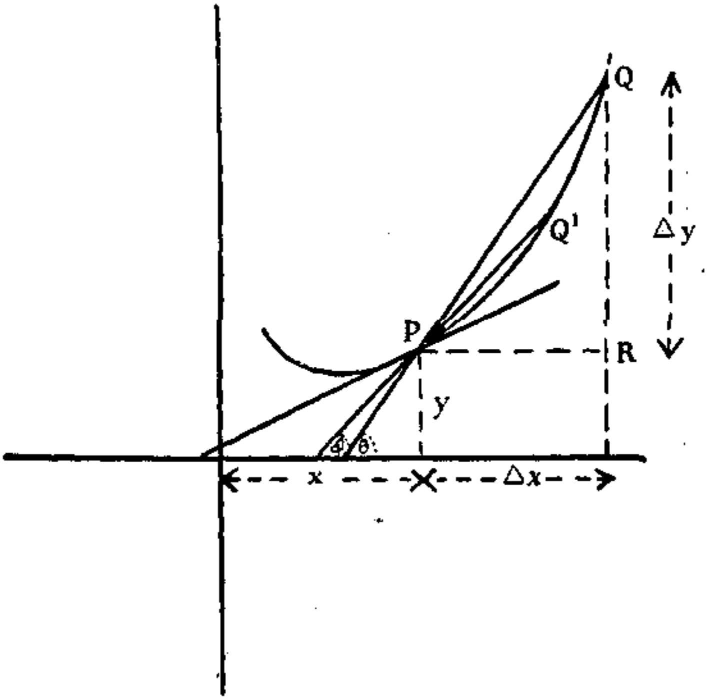

FIG. 91.

If Δx be made smaller, or Q moved nearer to P, say to Q′, then $\frac{\Delta y}{\Delta x} = \tan \phi$, which is the gradient of the chord PQ′. Similarly for any other position of Q.

Since $\frac{\Delta y}{\Delta x} = \tan \theta$ applies to the portion PQ as a whole, we may call its value the average gradient of the portion PQ of the curve.

Thus the average gradient of a curve lying between two points he gradient of the straight line joining those points.

Gradient of a Curve at a Point

If, now, Δx be made smaller and smaller—that is, if Q be :ought nearer and nearer to P—the chord will rotate about P and approach nearer and nearer to the tangent at P. Thus the tangent is the limiting position of the chord as Δx approaches zero value, and Q coincides with P.

But the average gradient of this exceedingly small portion of curve is the gradient of the line joining P and Q, which has become the tangent at P.

Thus the gradient of the curve at the point P is given by the gradient of the tangent drawn at P.

Thus the gradient of a curve at a point is the gradient of the tangent at the point.

10. Arithmetical Example

The following will serve to illustrate the behaviour of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

$$y = x^2$$

Let the initial value of $x = 3$.

Then the initial value of $y = 9$.

Now let x receive an increment Δx, and y, Δy.

The values of $\frac{\Delta y}{\Delta x}$ are tabulated for various values of Δx.

Initial Value of x.	New Value of x.	Increase in x Δx.	Initial Value of y.	New Value of y.	Increase in y Δy.	$\frac{\Delta y}{\Delta x}$
3	4	1	9	16	7	7
3	3·8	0·8	9	14·44	5·44	6·8
3	3·5	0·5	9	12·25	3·25	6·5
3	3·2	0·2	9	10·24	1·24	6·2
3	3·01	0·01	9	9·0601	0·0601	6·01
3	3·001	0·001	9	9·006001	0·006001	6·001
3	3·0001	0·0001	9	9·00060001	0·00060001	6·0001

From this we see that as Δx decreases, $\frac{\Delta y}{\Delta x}$ decreases; and by taking Δx small enough, we can bring $\frac{\Delta y}{\Delta x}$ as near to value 6 as we please.

$\therefore$ The limiting value of

$$\frac{\Delta y}{\Delta x} \longrightarrow 6 \text{ as } \Delta x \longrightarrow 0$$

Thus, in both cases, we find that *as Δx approaches zero value, $\frac{\Delta y}{\Delta x}$ approaches a limiting value, namely*

(1) *the gradient of the tangent (in the first case);*
(2) *the value 6 (in the second case).*

11. Velocity

A similar problem arises in the consideration of velocity in Mechanics.

Velocity (or speed) is defined as "distance travelled in unit time," being expressed in mls. per hr., ft. per sec., cms. per sec., etc. Perhaps speed would be the more correct term, since velocity has regard to direction as well as distance and time.

Since "distance travelled" implies a change in the distance of a moving body from some fixed arbitrary point, we may denote it by Δs. Similarly the time occupied by the movement implies an increase in time reckoning from a fixed starting point of time; we may therefore denote the time occupied by Δt.

Then since

$$\text{Velocity} = \frac{\text{distance travelled}}{\text{time occupied}}$$

we may write:

$$\text{Velocity, } v = \frac{\text{change in distance}}{\text{change in time}}$$

$$\therefore \quad v = \frac{\Delta s}{\Delta t}$$

2. Constant Velocity

If a train travels a distance of 40 mls. in 2 hrs., we say its velocity is 20 m.p.h., or $\frac{1}{3}$ ml. per min. In doing so, we are implying that its velocity is *constant* throughout the journey or that it travels *equal distances in equal times*. Thus in the above, v represents a constant velocity, if $\frac{\Delta s}{\Delta t}$ has the same value for all values of Δt.

13. Variable Velocity

In actual practice, however, owing to starting, stopping, curves and gradients of line, etc., the velocity of the train would vary throughout the journey: equal distances would not be covered in equal times and the velocity would be *variable*.

An analysis of the journey showing the actual position of the train at various times (as shown below) would enable us to find its actual velocity for separate portions of its journey.

Suppose the train starts at 3 p.m. and travels as shown.

Distance s (mls.)	0	5	10	15	20	25	30	35	40
Time p.m.	3 p.m.	3·18	3·33	3·49	4·10	4·24	4·36	4·47	5 p.m.

Time taken t (mins.)	18	15	16	21	14	12	11	13
Vel. $=\frac{\Delta s}{\Delta t}$ (mls. per min.)	$\frac{5}{18}=0{\cdot}27$	0·33	0·31	0·34	0·36	0·42	0·45	0·38

14. Average Velocity

As shown above, the velocity is clearly variable throughout. Also we must conclude that in all probability it varied *during* each single 5-mile run. The figures showing the velocity above could only express the *average* velocity during each 5 mls. Taking the 5 mls. between the twenty-fifth and thirtieth mile-posts and noting that the time occupied was 12 mins., we find

$$\frac{\Delta s}{\Delta t} = 0{\cdot}42 \text{ ml. per min.} = \textit{average} \text{ velocity for the 5 mls.}$$

Thus the average velocity throughout any period Δt is denoted by $\frac{\Delta s}{\Delta t}$.

$$\therefore \quad \frac{\Delta s}{\Delta t} = \textit{average velocity during the interval } \Delta t.$$

15. Velocity at a Point

If we wish to find the velocity with which the train passed a certain point (say the twenty-second mile post), the above table would merely state that the average velocity between the 20- and 25-mile-posts was 0·36 ml. per min., which cannot be taken as a true value of the velocity we require to know, owing to probable variations during the 5 mls.

Even if we knew the times at which the train passes the twenty-first and twenty-third mile-posts, respectively, we could not guarantee that the velocity obtained from our

calculation would be the actual velocity at the twenty-second mile-post, since there is a possibility of a change of velocity in the 2 mls. in question.

But if we knew the times at which it passed the points 100 yds. in front of, and 100 yds. beyond the post, and so calculated the velocity over the 200 yds. between, we should approach a truer estimate of the velocity at the post, since the probability of a change in the velocity is lessened by the smaller distance and time considered: the smaller the distance and time the nearer would the calculated velocity approach to the actual velocity.

The actual velocity at the point is the limiting value of the estimated velocity as the time interval approaches zero.

Thus,

$$\textit{Velocity at a point} = \textit{limiting value of } \frac{\Delta s}{\Delta t} \textit{ as } \Delta t \longrightarrow 0$$

Differentiation from First Principles

16. Gradient at a point on the curve, $y = x^2$

We have previously seen that the gradient at any point on a curve can be obtained by drawing a tangent to the curve at this point, and then the gradient of this tangent the gradient of the curve at the point.

This is, however, a cumbersome and not very accurate method and so we proceed to find a general algebraical way.

Considering $y = x^2$

Let Δx be an increment in x, and Δy the corresponding increment in y.

Then, as previously shown, replacing x by $x + \Delta x$ and y by $y + \Delta y$ in $y = x^2$ we have

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ &= x^2 + 2x \,.\, \Delta x + (\Delta x)^2 \end{aligned}$$

Subtracting $y = x^2$

we have $\Delta y = 2x \,.\, \Delta x + (\Delta x)^2$

This gives an expression for an increase in y—viz., Δy—in terms of the increase in x.

Dividing by Δx

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

This gives the ratio of corresponding increases in x and y.

Note that it also gives an expression for the gradient of the chord PQ (Fig. 91) for any value of Δx.

Now, if Δx be taken very small (in Fig. 91 as Q moves along the curve to P), then the ratio $\frac{\Delta y}{\Delta x}$ becomes nearer and nearer in value to $2x$.

Now we can make Δx as small as we like, and consequently $\frac{\Delta y}{\Delta x}$ approaches a limit $2x$ as Δx becomes infinitely small or the limit differs from $2x$ by as little as we please.

This can be expressed with our previous notation as follows:

$$\frac{\Delta y}{\Delta x} \longrightarrow 2x \text{ as } \Delta x \longrightarrow 0$$

This limiting value of the ratio $\frac{\Delta y}{\Delta x}$ is usually denoted by $\frac{dy}{dx}$. Using this notation, we can rewrite the above result as:

$$\frac{dy}{dx} = 2x$$

$\frac{dy}{dx}$ *is called the differential coefficient of y with respect to x,* and the process of obtaining it is called *differentiation.*

It should be carefully noted that

(1) *The Differential Coefficient expresses the rate of increase of y per unit increase in x.* It states how many times y is increasing with respect to x.

Thus, considering $y = x^2$, where $\frac{dy}{dx} = 2x$. When $x = 3$,

$\frac{dy}{dx} = 6$. At this point on the curve, or for this value of x, y is increasing six times as fast as x is increasing.

(2) It expresses *the gradient of the tangent to the curve representing the function* at any point on the curve. If θ be the slope of the curve at any point, then $\frac{dy}{dx} = \tan\theta$.

(3) *It must be regarded as one quantity.* It does not represent dy divided by dx, but is *the limit to which* $\frac{\Delta y}{\Delta x}$ *tends as* Δx *becomes indefinitely small.* The fractional form indicates that it is the limiting value of a fraction $\frac{\Delta y}{\Delta x}$.

17. A Geometrical Illustration

Fig. 92 represents a square metal plate ABCD of side x units.

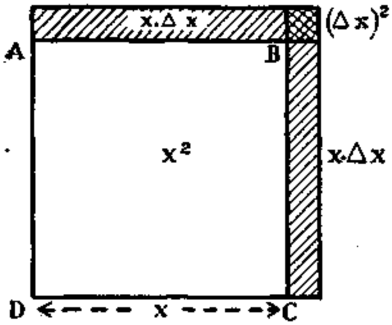

FIG. 92.

Let the area be y sq. units.

Then $$y = x^2$$

When heated, the side is increased in length by Δx.

Then the corresponding increase in area will be represented by Δy.

From the figure, it is seen that

$$\Delta y = 2x \,.\, \Delta x + (\Delta x)^2$$

$$\therefore \frac{\text{Increase in area}}{\text{Increase in side}} = \frac{2x \,.\, \Delta x + (\Delta x)^2}{\Delta x}$$

$$= 2x + \Delta x$$

It is clear, as before, that as Δx decreases, this ratio tends to become $2x$—*i.e.*, the ratio of the increase in area to the increase in the side ultimately becomes $2x$ when the increases are infinitely small.

18. Functions of the First Degree

The student has learnt that in the general equation of the first degree, the graph of which is a straight line, viz.

$$y = ax + b$$

b represents the intercept on the y axis and a represents the gradient of the line. Consequently, a is the differential coefficient of $ax + b$.

This can be found algebraically by the method used above, as follows:

Let Δx be an increment in x.

Let Δy be the corresponding increment in y.

Substituting in

$$y = ax + b$$

then $$y + \Delta y = a(x + \Delta x) + b$$

Subtracting $$\Delta y = a \,.\, \Delta x$$

$$\therefore \frac{\Delta y}{\Delta x} = a.$$

This is a constant—*i.e.*, it remains the same, however large or small the values of Δx and Δy.

$\therefore$ The differential coefficient is a

that is, if $$y = ax + b$$

$$\frac{dy}{dx} = a$$

Note on Constants.—It will be noted that in differentiating $y = ax + b$, the constant b disappears. It is clear that a system of parallel straight lines will have the same gradient but different values of b. Hence these cannot affect the result. In general, since a constant does not change, there can be no rate of change and its differential coefficient must be zero.

Examples

(1) If $y = 3x + 7$, $\frac{dy}{dx} = 3$.

(2) If $y = 2x - 6$, $\frac{dy}{dx} = 2$.

(3) If $y = 10 - x$, $\frac{dy}{dx} = -1$.

(4) If $y = \frac{x}{3}$, $\frac{dy}{ax} = \frac{1}{3}$.

(5) If $y = 5 - \frac{3x}{4}$, $\frac{dy}{dx} = -\frac{3}{4}$.

19. Functions of Higher Degree than the First

(1) Let $\mathbf{y = ax^2 + b}$, where a and b are constants.
Let Δx be an increment in x.
Then Δy is the corresponding increment in y.
Replacing x by $(x + \Delta x)$ and y by $(x + \Delta y)$ in $y = ax^2 + b$

$$\begin{aligned} y + \Delta y &= a(x + \Delta x)^2 + b \\ &= ax^2 + 2ax \, . \, \Delta x + a \, . \, (\Delta x)^2 + b \end{aligned}$$

But $\qquad y = ax^2 \qquad\qquad + b$

Subtracting, $\Delta y = 2ax \, . \, \Delta x + a \, . \, (\Delta x)^2$

Dividing by Δx

$$\frac{\Delta y}{\Delta x} = 2ax + a \, . \, \Delta x$$

$$\therefore \quad \frac{\Delta y}{\Delta x} \longrightarrow 2ax \text{ when } \Delta x \longrightarrow 0$$

$$\therefore \quad \text{In the limit, } \frac{dy}{dx} = 2ax.$$

We have seen that when

$$y = x^2, \frac{dy}{dx} = 2x$$

We now see that when

$$y = ax^2 + b, \frac{dy}{dx} = 2ax$$

i.e., the introduction of a, the *constant* coefficient of x^2, produces a differential coefficient a times as large as before.

The added constant (which of course may be positive or negative) disappears in the differential coefficient.

Examples

(1) If $y = 3x^2 + 2$, $(a = 3, b = 2)$

$$\therefore \frac{dy}{dx} = 2 \times 3 \times x = 6x.$$

(2) If $y = 5 - 2x^2$, $(a = -2, b = 5)$

$$\therefore \frac{dy}{dx} = 2 \times (-2) \times x = -4x.$$

(3) If $y = \frac{x^2}{4} - 7$, $(a = \frac{1}{4}, b = -7)$

$$\therefore \frac{dy}{dx} = 2 \times \tfrac{1}{4} \times x = \tfrac{1}{2}x.$$

(4) If $r = a\theta^2$.

$$\therefore \frac{dr}{d\theta} = 2a\theta.$$

(5) If $s = \frac{1}{2}ft^2$,

$$\therefore \frac{ds}{dt} = 2 \times \tfrac{1}{2} \times f \times t = ft.$$

(2) Let $$\mathbf{y = x^3}$$

Let Δx be an increment in x and Δy the corresponding increment in y.

Substituting $x + \Delta x$ for x, and $y + \Delta y$ for y in $y = x^3$.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^3 \\ &= x^3 + 3x^2 . \Delta x + 3x . (\Delta x)^2 + (\Delta x)^3 \end{aligned}$$

Subtracting $y = x^3$

$$\Delta y = 3x^2 . \Delta x + 3x . (\Delta x)^2 + (\Delta x)^3$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x \,.\, \Delta x + (\Delta x)^2$$

$$\therefore \quad \frac{\Delta y}{\Delta x} \longrightarrow 3x^2 \text{ when } \Delta x \longrightarrow 0$$

$$\therefore \qquad \text{in the limit } \frac{dy}{dx} = 3x^2.$$

(3) Let $\mathbf{y = ax^3 + b}$ (where a and b are constants).

Let Δx be an increment in x and Δy the corresponding increment in y.

Proceeding as in previous cases,

$$\begin{aligned} y + \Delta y &= a(x + \Delta x)^3 + b \\ &= ax^3 + 3ax^2 \,.\, \Delta x + 3ax \,.\, (\Delta x)^2 + a(\Delta x)^3 + b. \end{aligned}$$

Subtracting $y = ax^3 + b$

$$\Delta y = 3ax^2 \,.\, \Delta x + 3ax \,.\, (\Delta x)^2 + a(\Delta x)^3$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 3ax^2 \,. + 3ax \,.\, \Delta x + a(\Delta x)^2$$

Proceeding to the limit when $\Delta x \longrightarrow 0$

$$\frac{dy}{dx} = 3ax^2$$

It will be noted that the introduction of a and b is followed by the same result as in No. (1) (p. 247).

Examples

(1) If $y = 6x^3 + 7$,

$$\therefore \qquad \frac{dy}{dx} = 3 \times 6 \times x^2 = 18x^2.$$

(2) If $y = 10 - 2x^3$,

$$\therefore \qquad \frac{dy}{dx} = 3 \times (-2) \times x^2 = -6x^2.$$

(3) If $V = 5l^3$,

$$\therefore \qquad \frac{dV}{dl} = 3 \times 5 \times l^2 = 15l^2.$$

(4) If $s = \frac{3}{4}t^3 - 9$,

$$\therefore \qquad \frac{ds}{dt} = 3 \times \tfrac{3}{4} \times t^2 = \frac{9t^2}{4}.$$

(4) If $l = 2\theta^3$,

$$\therefore \qquad \frac{dl}{d\theta} = 3 \times 2 \times \theta^2 = 6\theta^2.$$

(4) **General Case.** Let $y = x^n$ (where n is any positive integral constant).

Let Δx denote an increment in x and Δy the corresponding increment in y.

Substituting $x + \Delta x$ for x, and $y + \Delta y$ for y in $y = x^n$

$$y + \Delta y = (x + \Delta x)^n$$

Expanding by the Binomial Theorem (see p. 229) we have

$$y + \Delta y = x^n + nx^{n-1}.\Delta x + \frac{n(n-1)}{\underline{|2}}x^{n-2}(\Delta x)^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}(\Delta x)^3 + \ldots$$

But $y = x^n$

Subtracting

$$\Delta y = nx^{n-1}.\Delta x + \frac{n(n-1)}{\underline{|2}}x^{n-2}.(\Delta x)^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}(\Delta x)^3 + \ldots$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = n.x^{n-1} + \frac{n(n-1)}{\underline{|2}}x^{n-2}.(\Delta x) + \frac{n(n-1)(n-2)}{\underline{|3}}x^{n-3}(\Delta x)^2 + \ldots$$

$$\therefore \quad \frac{\Delta y}{\Delta x} \longrightarrow n.x^{n-1} \text{ as } \Delta x \longrightarrow 0,$$

since every term after the first contains some power of Δx as a factor

$$\therefore \quad \frac{dy}{dx} = n \,.\, x^{n-1}.$$

By analogy with the previous example, it will be clear that if

$$y = ax^n, \quad \frac{dy}{dx} = nax^{n-1}$$

20. Rule for Differentiating, $y = ax^n$

By collecting and examining our results, we see

If $\qquad y = ax^2 + b, \ \dfrac{dy}{dx} = 2ax$

If $\qquad y = ax^3 + b, \ \dfrac{dy}{dx} = 3ax^2$

If $\qquad y = ax^n, \qquad \dfrac{dy}{dx} = nax^{n-1}$

Thus a simple rule for differentiating expressions of this type may be stated as follows :

Multiply the coefficient of the x term by its index for the new coefficient, and reduce the index by 1 for the new index.

Examples

(1) If $y = 7x^3 + 3 \qquad \dfrac{dy}{dx} = 3 \times 7 \times x^{3-1} = 21x^2.$

(2) If $y = 5 - 6x^5 \qquad \dfrac{dy}{dx} = 5 \times (-6) \times x^{5-1} = -30x^4.$

(3) If $r = 6p^2 \qquad \dfrac{dr}{dp} = 2 \times 6 \times p^{2-1} = 12p.$

(4) If $s = 20 + \dfrac{3t^4}{4} \qquad \dfrac{ds}{dt} = 4 \times \frac{3}{4} \times t^{4-1} = 3t^3.$

(5) If $y = \dfrac{x^5}{2} \qquad \dfrac{dy}{dx} = 5 \times \frac{1}{2} \times x^{5-1} = \frac{5}{2}x^4.$

The student will later learn to prove, by the Binomial Theorem, that this result—*i.e.*, that the differential coefficient of $ax^n = nax^{n-1}$—is also true for cases in which n is fractional or negative. As a verification of this in a particular case, we will differentiate

$$y = ax^{-2} \quad \text{or,} \quad y = \frac{a}{x^2}$$

Let Δx be the increment in x and Δy the corresponding increment in y.

Substituting as before

$$y + \Delta y = a(x + \Delta x)^{-2}$$
$$= \frac{a}{(x + \Delta x)^2}$$

Subtracting,

$$\Delta y = \frac{a}{(x + \Delta x)^2} - \frac{a}{x^2}$$
$$= \frac{ax^2 - (x + \Delta x)^2 \cdot a}{(x^2)(x + \Delta x)^2}$$
$$= \frac{ax^2 - ax^2 - 2ax \cdot \Delta x - a(\Delta x)^2}{(x^2)(x + \Delta x)^2}$$
$$= \frac{-2ax \cdot \Delta x - a(\Delta x)^2}{(x^2)(x + \Delta x)^2}$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{-2ax - a \cdot \Delta x}{(x^2)(x + \Delta x)^2}$$

Proceeding to the limit

$$\frac{\Delta y}{\Delta x} \longrightarrow \frac{-2ax}{(x^2)(x^2)} \text{ when } \Delta x \longrightarrow 0$$
$$\therefore \quad \frac{dy}{dx} = -\frac{2ax}{x^4}$$
$$\therefore \quad \frac{dy}{dx} = -\frac{2a}{x^3}$$

It will be found on examination that this result can be obtained by applying the rule given above. It is beyond the

scope of this chapter to prove that it also holds when the index is fractional: but we shall assume that if

$$y = ax^n, \ \frac{dy}{dx} = nax^{n-1}$$

in all cases, even when n is negative or fractional.

Examples

(1) $y = \frac{3}{x^2} = 3x^{-2}$. $\therefore \ \frac{dy}{dx} = -6x^{-3} = -\frac{6}{x^3}$.

(2) $y = \sqrt{x} = x^{\frac{1}{2}}$. $\therefore \ \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

(3) $y = 0{\cdot}2x^{1{\cdot}26}$. $\therefore \ \frac{dy}{dx} = 0{\cdot}252x^{0{\cdot}26}$.

(4) $y = 4\sqrt[3]{x^2} = 4x^{\frac{2}{3}}$. $\therefore \ \frac{dy}{dx} = \frac{8}{3}x^{-\frac{1}{3}} = \frac{8}{3\sqrt[3]{x}}$.

(5) $y = \frac{6}{x} = 6x^{-1}$. $\therefore \ \frac{dy}{dx} = -6x^{-2} = -\frac{6}{x^2}$.

21. Differentiation of a Sum

Let $$y = ax^2 + bx + c$$

Let Δx be an increment in x and Δy the corresponding increment in y.

Substituting $(x + \Delta x)$ for x, and $(y + \Delta y)$ for y, in

$$y = ax^2 + bx + c$$

we have $y + \Delta y = a(x + \Delta x)^2 + b(x + \Delta x) + c$

i.e., $y + \Delta y = ax^2 + 2ax \, . \, \Delta x + a(\Delta x)^2 + bx + b \, . \, \Delta x + c$

Subtracting, $\Delta y = 2ax \, . \, \Delta x + a(\Delta x)^2 + b \, . \, \Delta x$.

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 2ax + a(\Delta x) + b$$

$$\therefore \ \frac{\Delta y}{\Delta x} \longrightarrow 2ax + b \quad \text{when} \quad \Delta x \longrightarrow 0$$

$$\therefore \ \frac{dy}{dx} = 2ax + b$$

But $2ax$ is the differential coefficient of ax^2 and b the differential coefficient of bx and the differential coefficient of c is zero.

Thus, from this and similar examples we may deduce the rule that *the differential coefficient of a sum of functions is the sum of the differential coefficients of the separate functions.*

This will be proved in Vol. III.

Examples

(1) If $y = 2x^3 - 3x + 6$ $\qquad \dfrac{dy}{dx} = 6x^2 - 3.$

(2) If $s = t^2 + 3t - 4$ $\qquad \dfrac{ds}{dt} = 2t + 3.$

(3) If $y = x^2 + \dfrac{2}{\sqrt{x}}$ $\qquad \dfrac{dy}{dx} = 2x - \dfrac{1}{\sqrt{x^3}}.$

(4) If $v = u^{1\cdot2} - \dfrac{1}{2u^{2\cdot5}}$ $\qquad \dfrac{dv}{du} = 1{\cdot}2u^{0\cdot2} + \dfrac{2{\cdot}5u^{-3\cdot5}}{2}$

$$= 1{\cdot}2u^{0\cdot2} + \frac{1{\cdot}25}{u^{3\cdot5}}.$$

22. Notation for Differential Coefficient

The differential coefficient of y with respect to x may be expressed in various ways: (*a*) $\dfrac{dy}{dx}$, (*b*) $f'(x)$, (*c*) $\dfrac{d}{dx}\{f(x)\}$, (*d*) Dy, (*e*) y'.

Thus $\dfrac{d}{dx}(2x^3 + 3)$ presents the same problem as:

If $y = 2x^2 + 3$, find $\dfrac{dy}{dx}$.

$$\therefore \quad \frac{d}{dx}(2x^2 + 3) = 4x.$$

Similarly,

$$\frac{d}{dz}\left(2z^{0\cdot4} - \frac{4}{z^{1\cdot2}}\right) = 0{\cdot}8z^{-0\cdot6} + 4{\cdot}8z^{-2\cdot2} = \frac{0{\cdot}8}{z^{0\cdot6}} + \frac{4{\cdot}8}{z^{2\cdot2}}.$$

Practical Examples

We now proceed to apply the foregoing results to some practical problems.

1. Velocity and Acceleration

Velocity

Suppose a body moves s ft. in t secs. where

$$s = ut + \tfrac{1}{2}ft^2 \text{ (u and f being constants)}$$

Let Δt be an increment in t and Δs the corresponding increment in s.

Substituting $(t + \Delta t)$ for t, and $(s + \Delta s)$ for s in

$$s = ut + \tfrac{1}{2}ft^2$$

we have

$$\begin{aligned} s + \Delta s &= u(t + \Delta t) + \tfrac{1}{2}f(t + \Delta t)^2 \\ &= ut + u \,.\, \Delta t + \tfrac{1}{2}ft^2 + ft \,.\, \Delta t + \tfrac{1}{2}f(\Delta t)^2 \end{aligned}$$

Subtracting, $\Delta s = u \,.\, \Delta t + ft \,.\, \Delta t + \tfrac{1}{2}f(\Delta t)^2$

This is an expression for the displacement Δs during the interval of time Δt at the end of t secs.

Dividing by Δt,

$$\frac{\Delta s}{\Delta t} = u + ft + \tfrac{1}{2}f \,.\, \Delta t$$

This is an expression for the average velocity during the interval Δt secs. at the end of t secs.

Then $$\frac{\Delta s}{\Delta t} \longrightarrow u + ft, \text{ as } \Delta t \longrightarrow 0$$

and in the limit,

$$\frac{ds}{dt} = u + ft.$$

But the limiting value of $\dfrac{\Delta s}{\Delta t}$ is the velocity at a point

$$\therefore \quad \frac{ds}{dt} = v = u + ft.$$

This gives the value of the velocity at the instant t secs. from the start.

Acceleration

Since acceleration means a change of velocity with respect to time, just as velocity means a change of distance with respect to time, it is obvious that we shall obtain an expression for acceleration from that for velocity, by applying the same process to the velocity expression as we did to the distance expression.

Since $$v = \frac{ds}{dt}$$

$$acceleration = \frac{dv}{dt}.$$

To find the acceleration, we differentiate s for the velocity, and then differentiate the result, in both cases with respect to t. We have therefore differentiated the distance expression, or s, twice. The second result is called the *second differential coefficient* and is written $\frac{d^2s}{dt^2}$

It is usually read—" d two s over dt squared."

Applying this to the previous example, in which we found

$$\frac{ds}{dt} = v = u + ft$$

we can now write

$$acceleration = \frac{d^2s}{dt^2} = \frac{dv}{dt} = f$$

Thus we see the relation between the distance, velocity and acceleration in the equations so well known to students of Mechanics.

(a) $s = ut + \frac{1}{2}ft.^2$

(b) $v = \frac{ds}{dt} = u + ft.$

(c) acceleration $= f = \frac{d^2s}{dt^2}.$

2. *Find the gradient of the graph of*

$$y = 4x^2 - 3x + 2$$

when x is (a) 2, (b) $-$ 1.

The gradient at a point is expressed by $\frac{dy}{dx}$

If $\quad y = 4x^2 - 3x + 2$

$$\frac{dy}{dx} = 8x - 3$$

$\therefore$ If $\quad x = 2, \frac{dy}{dx} = 13$

If $\quad x = -1, \frac{dy}{dx} = -11$

Thus the gradient when $x = 2$ is 13 and the gradient when $x = -1$ is -11.

3. *What is the value of $7 - 2x + 3x^2$ when its rate of increase with respect to x is 4?*

Let $\quad y = 7 - 2x + 3x^2$

Then $\quad \frac{dy}{dx} = -2 + 6x.$

This is the rate of increase of y with respect to x.

$$\therefore \quad -2 + 6x = 4$$
$$6x = 6$$
and $$x = 1$$

Then the value of $7 - 2x + 3x^2$ when $x = 1$

is $7 - 2 + 3 = 8$.

4. *A spherical balloon is being blown up and its radius is increasing at the rate of 0·25 in. per sec. At what rate is its surface increasing when the radius is 2·5 ins.?*

The surface area $A = 4\pi r^2$ (where r = radius)

$$\therefore \quad \frac{dA}{dr} = 8\pi r$$

$\therefore$ The rate of increase of the area A with respect to r is $8\pi r$—that is, the area is increasing $8\pi r$ times as fast as the radius. But the radius is increasing at the rate of 0·25 in. per sec.

$\therefore$ The area A is increasing at the rate of $8\pi r \times 0{\cdot}25$ sq. in. per sec.

But $$r = 2{\cdot}5 \text{ ins.}$$

$\therefore$ The rate of increase of A

$$= (8\pi \times 2{\cdot}5 \times 0{\cdot}25) \text{ sq. in. per sec.}$$
$$= \mathit{15{\cdot}7 \ sq. \ ins. \ per \ sec.}$$

5. *A body moves along a straight line in such a way that its distance s ft. from its starting-point at the end of t secs. is given by* $s = 3t^2 - 2t + 5$.

Find

(a) *the distance it travels between* 3 *and* 3·1 *secs. from the start;*

(b) *its average speed during the same interval;*

(c) *its velocity when* $t = 3$ *secs.;*

(d) *its acceleration.*

(a) *Displacement* (Δs) *between* 3 *and* 3·1 *secs.*

$$s = 3t^2 - 2t + 5$$

Let Δt represent an increment in t and Δs the corresponding increment in s.

Substituting $(t + \Delta t)$ and $(s + \Delta s)$ for t and s respectively in the given equation we have

$$\begin{aligned} s + \Delta s &= 3(t + \Delta t)^2 - 2(t + \Delta t) + 5 \\ &= 3t^2 + 6t \,.\, \Delta t + 3(\Delta t)^2 - 2t - 2\Delta t + 5 \end{aligned}$$

Subtracting,

$$\Delta s = 6t \,.\, \Delta t + 3(\Delta t)^2 - 2\Delta t.$$

This is the general expression for a displacement Δs during an interval Δt measured at the end of t secs.

$\therefore$ When $t = 3$ secs. and $\Delta t = 0{\cdot}1$ sec.

$$\begin{aligned} \Delta s &= (6 \times 3 \times 0{\cdot}1) + 3(0{\cdot}1)^2 - (2 \times 0{\cdot}1) \\ &= 1{\cdot}8 + 0{\cdot}03 - 0{\cdot}2 \\ &= \mathit{1{\cdot}63 \ ft.} \end{aligned}$$

As a check on this we may proceed thus

$$s = 3t^2 - 2t + 5$$

$\therefore$ Distance in 3 secs. $= 27 - 6 + 5 = 26$ ft.

and distance in 3·1 secs. $= 3(3{\cdot}1)^2 - 2(3{\cdot}1) + 5$

$= 27{\cdot}63$ ft.

$\therefore$ Distance in interval between 3 and 3·1 secs.

$= $ *1·63 ft.*

(b) *Average Speed during the interval*

$\Delta s = 6t \,.\, \Delta t + 3(\Delta t)^2 - 2\Delta t$ (as above).

Dividing by Δt,

$$\frac{\Delta s}{\Delta t} = \text{average speed} = 6t + 3 \,.\, \Delta t - 2$$

$\therefore$ When $t = 3$ and $\Delta t = 0{\cdot}1$

Average speed $= 18 + 0{\cdot}3 - 2$

$= $ *16·3 ft. per sec.*

As a check on this, we may proceed thus

Distance travelled between 3 and 3·1 secs. $= 1{\cdot}63$ ft. (see above).

i.e., Distance in 0·1 sec. $= 1{\cdot}63$ ft.

$$\therefore \quad \text{Average speed} = \frac{1{\cdot}63}{0{\cdot}1} = \textit{16·3 ft. per sec.}$$

(c) *Velocity when* $t = 3$ *secs.*

Since $s = 3t^2 - 2t + 5$

$$v = \frac{ds}{dt} = 6t - 2$$

$\therefore$ When $t = 3$, $v = 18 - 2 = $ *16 ft. per sec.*

This could also have been obtained from

$$\frac{\Delta s}{\Delta t} = 6t + 3\Delta t - 2 \quad \text{(see above)}$$

since velocity at a point $=$ limit of $\frac{\Delta s}{\Delta t}$ when $\Delta t \to 0$

$$\therefore \quad v = \frac{ds}{dt} = 6t - 2 \text{ where } t = 3$$

$$\therefore \quad v = \textit{16 ft. per sec.}$$

(d) *Acceleration.*

Acceleration is denoted by $\frac{d^2s}{dt^2}$

$$s = 3t^2 - 2t + 5$$

$$\therefore \quad \frac{ds}{dt} = 6t - 2$$

$$\therefore \quad \frac{d^2s}{dt^2} = 6 \textit{ ft. per sec. per sec.}$$

A check on this may be obtained as follows:

Since velocity = limit of $6t + 3\Delta t - 2$ (see above).

When $t = 3{\cdot}1$ secs. and $\Delta t = 0$, velocity = $(6 \times 3{\cdot}1) - 2 = 16{\cdot}6$ ft. per sec.

When $t = 3$ secs. and $\Delta t = 0$, velocity = $(6 \times 3) - 2 = 16$ ft. per sec.

$\therefore$ Velocity has increased by $0{\cdot}6$ ft. per sec. in $0{\cdot}1$ sec.

$\therefore$ Acceleration $= \frac{0{\cdot}6}{0{\cdot}1} = 6$ *ft. per sec.*^{2}.

6. *A body moves s ft. in t secs. in accordance with the equation*

$$s = t^3 - 2t^2 + t + 3$$

Find when its velocity is 10 ft. per sec. and when its acceleration is 20 ft. per sec.^{2}

$$s = t^3 - 2t^2 + t + 3$$

$$\therefore \quad \text{velocity} = \frac{ds}{dt} = 3t^2 - 4t + 1$$

$\therefore$ when the velocity is 10 ft. per sec.

$$3t^2 - 4t + 1 = 10$$

which gives

$$t = 2{\cdot}52 \text{ or } -1{\cdot}19 \text{ secs.}$$

(the latter answer being inadmissible).

$\therefore$ $v = 10$ ft. per sec. when $t = 2{\cdot}52$ *secs.*

$$\text{acceleration} = \frac{d^2s}{dt^2} = 6t - 4$$

$\therefore$ When the acceleration is 20 ft. per sec.^{2}

$$6t - 4 = 20$$
$$\therefore \qquad t = 4 \textit{ secs.}$$

7. *A wheel rotates through* θ *radians in* t *secs., where*

$$\theta = 20 + 15t - t^2.$$

Find

(a) *the number of radians turned through in* 6 *secs.;*

(b) *the angular velocity in radians per sec. at the end of* 4 *secs.;*

(c) *its initial angular velocity;*

(d) *its acceleration in radians per sec. per sec.;*

(e) *at what time it will come to rest.*

(*a*) $\theta = 20 + 15t - t^2$

$\therefore$ When $t = 6$ secs.

$$\theta = 20 + 90 - 36 = \textit{74 radians.}$$

(*b*) Angular velocity (ω)

$$= \frac{d\theta}{dt}$$
$$= 15 - 2t$$

$\therefore$ When $t = 4$, $\omega = 15 - 8 =$ *7 radians per sec.*

(*c*) Initial angular velocity = velocity when $t = 0$

$$\omega = 15 - 2t \text{ (see above)}$$

$\therefore$ When $t = 0$, initial angular velocity = *15 radians per sec.*

(*d*) Acceleration = change in velocity with respect to time

$$= \frac{d\omega}{dt}\left(= \frac{d^2\theta}{dt^2}\right)$$
$$\omega = 15 - 2t$$
$$\therefore \quad \frac{d\omega}{dt} = -2$$

$\therefore$ Acceleration = *− 2 radians per sec. per sec.*

(*e*) Time when it comes to rest = time when $\omega = 0$

But $\omega = 15 - 2t$

$\therefore \quad \omega = 0$ when $t = 7\frac{1}{2}$ *secs.*

EXERCISE 19

1. Differentiate from first principles:

(a) $y = 2x + 3$
(b) $s = 5t^2 - 2t + 3.$
(c) $y = 4t^3 + t^2.$
(d) $y = \frac{1}{x^3}.$

2. Differentiate by inspection:

(a) $y = 5x^4 + 3x - 2.$
(b) $s = 6t^3 - 3t^2 + t + 5.$
(c) $f(x) = 3\sqrt{x}.$
(d) $s = r\theta$ (where r is a constant).
(e) $y = \frac{2x^2}{3} - \frac{5x}{6} + 7 - \frac{2}{x} + \frac{4}{x^2}.$
(f) $y = \sqrt{x} + \frac{1}{\sqrt{x}}.$
(g) $y = 2{\cdot}3x^{1{\cdot}2} - 2\sqrt{x} + \frac{7}{x}.$
(h) $\theta = 18 + 10t - 3t^2.$
(i) $y = 0{\cdot}3x^{1{\cdot}35}.$

3. Find the value of:

(a) $\frac{d}{du}(3u^2 + u - 5).$
(b) $\frac{d}{dx}\left(x^{1{\cdot}2} - \frac{1}{x^{0{\cdot}2}} + 2x^{1{\cdot}5}\right).$
(c) $\frac{dp}{dv}$ if $pv^{1{\cdot}4} = 400.$
(d) $\frac{dh}{dv}$ if $h = 0{\cdot}075v^{1{\cdot}5}.$
(e) $\frac{d}{dh}\left(0{\cdot}4 + \frac{0{\cdot}001}{h}\right).$

4. Find the value of $\frac{d^2s}{dt^2} + 3\frac{ds}{dt}$ when $s = 6t^3 - 5t^2 + 8t.$

5. Evaluate $\frac{2d^2y}{dx^2} + \frac{dy}{dx}$ when $y = 5 + \frac{3}{x}.$

6. Find $\frac{d^3s}{dt^3}$ when $s = t^4 - 3t^3 + 2t - 5$.

7. If $y = x^2$, where x is the side of a square, what is the meaning of $\frac{dy}{dx}$? At what rate is the area increasing when $x = 3\frac{1}{2}$ ins. and is growing at the rate of 0·01 in. per sec.?

8. A spherical toy-balloon is being blown up and its radius is increasing at the rate of 0·2 in. per sec. At what rate is its volume increasing when the radius is 3 ins.?

9. Find the side of a square whose area is increasing at the rate of 16 sq. ins. per in. increase in its side.

10. Draw the graph of $6 - 2x^2$ for values of x between -3 and $+3$. Draw, as exactly as possible, the tangents at the points where x is $-1·5$ and $+2$. Find the gradient of each tangent by using the scales on the axes, and compare the results with those obtained by differentiation.

11. Draw the graph of $\frac{x^2}{2} + 5$ between $x = -3$ and $x = +3$. Find, by drawing the tangents, the gradient of the curve at points whose abscissæ (*i.e.*, x values) are -2 and $+1$. Check your answers by differentiation.

12. Find the gradient of the curve $y = 3x^3 - 2x + 7$ at the points where $x = 1$ and $-0·5$.

13. For what values of x is the gradient of the curve $x^3 + \frac{x^2}{2} - 2x + 6$ equal to (*a*) 0, (*b*) -2?

14. Find the gradient of $y = 6x^2 - \frac{5}{x}$ when $x = 0·9$.

15. For what values of x is the gradient of the graph of $6 + 5x - x^2$ equal to (*a*) -2, (*b*) 0, (*c*) 1·5?

16. What is the value of the function $8 - 4x + \frac{x^2}{3}$ when its rate of increase with respect to x is 6?

17. If $s = 20 + 100t - 4t^2$, where s ft. is the space

covered by a moving body in t secs., what is the speed acquired in (*a*) t secs., (*b*) 4 secs.?

Show that the acceleration is constant.

18. A body moves s ft. in t secs., where

$$s = 10 + 5t + 12t^2 - t^3$$

Find

(*a*) its speed at the end of 2 secs.;
(*b*) its acceleration at the end of 3 secs.;
(*c*) when its acceleration is zero;
(*d*) when its speed is zero.

19. Find the speed and acceleration of a body at the end of 5 [illegible]s., if its equation of motion is

$$s = 3{\cdot}1 - 5t + 6t^2$$

[illegible]y moves s ft. in t secs. in accordance with the

$$s = 10 + 6t + 13t^2 - t^3$$

[illegible]

(*a*) its speed at the end of 3 secs.;
(*b*) when its speed is zero;
(*c*) its acceleration at the end of 2 secs.;
(*d*) when its acceleration is zero.

21. A body is thrown vertically upwards with a speed of 120 ft. per sec. Its height s ft. after t secs. is given by

$$s = 120t - 16t^2$$

(*a*) Find the distance travelled during the interval between $t = 2{\cdot}5$ and $t = 3$ secs.: hence find the average speed during that interval.

(*b*) Find a formula for its average speed during any interval, Δt, measured from the instant t secs. from the start.

(*c*) Use (*b*) to verify (*a*).

(*d*) Find a formula for the speed at any instant and hence find the speed when $t = 2$ secs.

(*e*) Find the time when the speed is zero, and the corresponding height of the body (*i.e.*, its maximum height).

22. A body is thrown upwards and its equation of motion is

$$s = 100t - 16t^2$$

Find

(*a*) when it reaches its greatest height;

(*b*) its speed when it is halfway up, and halfway down.

23. A wheel rotates through θ radians in t secs., where

$$\theta = 30 + 12t - t^2$$

Find

(*a*) the number of radians through which it rotates in 10 secs.;

(*b*) its angular velocity, ω, in radians per sec. at the end of 5 secs.;

(*c*) its initial angular velocity;

(*d*) its acceleration in radians per sec. per sec.;

(*e*) at what time it will come to rest.

24. Obtain, in terms of δx, the slope of the chord joining the points on the graph of the function $3x^2 - 4x + 7$ at which the values of x are 2·5 and $(2{\cdot}5 + \delta x)$. Calculate also the slope of the chord joining the points, on the same graph, at which the values of x are $(2{\cdot}5 - \delta x)$ and $(2{\cdot}5 + \delta x)$. (N.C.T.E.C., 1935.)

25. A beam is supported at each end and the values of the Bending Moment, M (tons-ft.) at distances x (ft.) from one end of the beam are given:

M . . .	0	6·3	9·6	10·0	9·6	6·3	0
x . . .	0	4	8	10	12	16	20

Plot M vertically and x horizontally, and draw a smooth curve through the points obtained. The shear force F (tons) at any distance x is given by the slope of the curve at this distance x. Find F for values of x equal to 6, 10 and 14 ft. (U.E.I., 1932.)

26. A body moves so that the distance s (ft.) covered in time t (secs.) from a certain instant is given by :

$$s = 0{\cdot}2t^2 + 10{\cdot}4$$

By means of the calculus, derive an expression for the velocity of the body. Calculate :

(1) The time that has elapsed when the body has travelled 100 ft.

(2) The velocity of the body when it has covered 100 ft. (U.E.I., 1935.)

27. A curve is represented by the following equation :

$$y = x^3 - 2x^2 - 11x + 12$$

Find the value of $\frac{dy}{dx}$ and give its value when $x = -2$.

(U.L.C.I., 1936.)

28. (*a*) The distance s ft. moved by a body in time t secs. is given by the equation $s = 4t^2$.

Calculate s when $t = 3$ and when $t = (3 + \delta t)$.

Hence obtain an expression for $\frac{\delta s}{\delta t}$, the average speed during the interval. State the speed at any instant, and give its value when $t = 3$ secs.

(*b*) Find an expression for the slope at any point of the curve represented by the equation :

$$y = 2x^3 - 9x^2 - 60x - 25$$

and give its value when $x = 5$. (U.L.C.I., 1935.)

29. The distance s (ft.) moved through by a body in time t (secs.) is given by the expression

$$s = 200t - 16t^2$$

Find the velocity $\left(\frac{ds}{dt}\right)$ at $t = 0$ and at $t = 4$ secs.

When will the body be at rest? (U.E.I.)

30. By means of a diagram illustrate what is meant by $\frac{dm}{dn}$, where m is any function of n.

If $m = 3n^2 - 17n - 2$ find $\frac{dm}{dn}$ when $n = 2$, and explain carefully what is meant by the result. (U.E.I.)

CHAPTER 15

MAXIMA AND MINIMA

1. The Sign of the Differential Coefficient—Stationary Values

We have seen that if $y = x^2$

$$\frac{dy}{dx} = 2x.$$

This is true for all values of x.

Hence if x is positive, $\frac{dy}{dx}$ is positive.

If x is negative, $\frac{dy}{dx}$ is negative.

If x is 0, $\frac{dy}{dx}$ is 0.

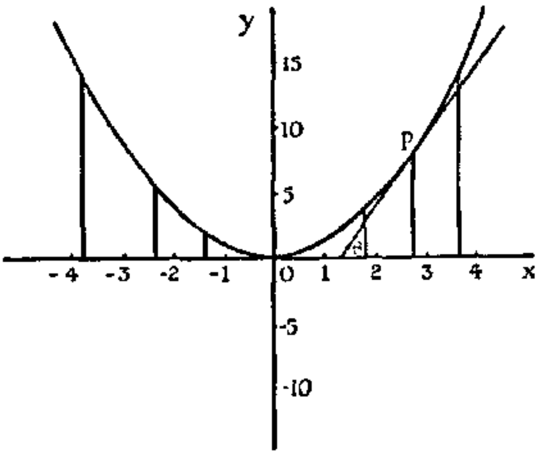

FIG. 93.

Let us examine these by the help of the graph of $y = x^2$ (Fig. 93).

First the student should recall the convention used in the representation of numbers when drawing a graph.

Numbers which are shown on the selected scale on the x axis are regarded as *increasing* from an infinite distance on the left or negative side of the origin, through zero, to an infinite distance on the right, or positive, side.

Similarly on the y axis the numbers are regarded as *increasing* from $-\infty$ below the x axis to $+\infty$ above the x axis.

If from points such as P on the curve of $y = x^2$ (Fig. 93) perpendiculars are drawn to the x axis, the lengths of these perpendiculars represent the values of x^2 or y at these points. It is evident that

> *as x increases from $-\infty$ to 0, the values of y are decreasing.*

At the origin the value of y is zero. At that point the curve touches the x axis which is a tangent to it and momentarily

> *the value of y is stationary,*

i.e., it is neither increasing nor decreasing.

Similarly,

> *as x increases from 0 to $+\infty$, the values of y are increasing.*

Comparing these facts with the sign of the differential coefficient, stated above:

(1) When x is negative,

$$\frac{dy}{dx} \textit{ is negative, and } y \textit{ is decreasing.}$$

(2) When x is positive,

$$\frac{dy}{dx} \textit{ is positive, and } y \textit{ is increasing.}$$

(3) When x is 0,

$$\frac{dy}{dx} \textit{ is } 0, \textit{ and } y \textit{ is stationary.}$$

It should also be noted that when x is positive the angle of slope, θ, is an acute angle, and tan θ, which is $\frac{dy}{dx}$, is positive.

When x is negative, θ is clearly an obtuse angle and tan θ is negative.

We conclude therefore that

(1) If $\frac{dy}{dx}$ is positive,

y is increasing as x is increasing.

(2) If $\frac{dy}{dx}$ is negative,

y is decreasing as x is increasing.

(3) If $\frac{dy}{dx}$ is 0,

y is stationary.

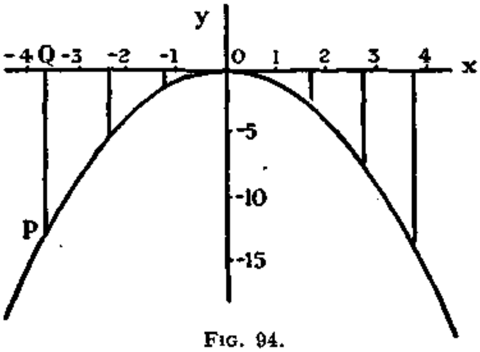

FIG. 94.

As another example we might consider the curve of

$$y = -x^2 \quad \text{(Fig. 94)}$$

Differentiating,

$$\frac{dy}{dx} = -2x$$

When x is negative, $\frac{dy}{dx}$ is positive.

Considering the lengths of the perpendiculars such as PQ (Fig. 94) for these negative values of x, it is seen that they are decreasing in length as x increases, but as they represent negative numbers, their absolute value is increasing; for example, -1 is greater than -5. Hence as x is increasing for negative value

$\frac{dy}{dx}$ *is positive and y is increasing.*

Similarly, for positive increasing values of x,

$\frac{dy}{dx}$ *is negative and y is decreasing.*

At the origin, as before,

$\frac{dy}{dx}$ *is zero, and y has a stationary value.*

It will readily be seen that similar results will be obtained in the cases of other functions and so we may conclude generally that

when y is a function of x,

(1) *if y is increasing as x increases,* $\frac{dy}{dx}$ *is positive;*

(2) *if y is decreasing as x increases,* $\frac{dy}{dx}$ *is negative;*

(3) *if* $\frac{dy}{dx}$ *is zero, y is stationary.*

2. Turning Points

Comparing the stationary points in the curves of

$$y = x^2 \quad \text{and} \quad y = -x^2$$

we see that there are important differences.

In the case of $y = x^2$, we note

(1) *The curve is changing direction:* the slope θ is changing from an *obtuse angle, through* 0, *to an acute angle.*

(2) The values of x^2 are *decreasing before* the stationary point and *increasing after.*

But *in* $y = -x^2$, the opposite of these is happening.

(1) *The curve is changing direction,* but θ is changing from an *acute angle, through* 0, *to an obtuse angle.*

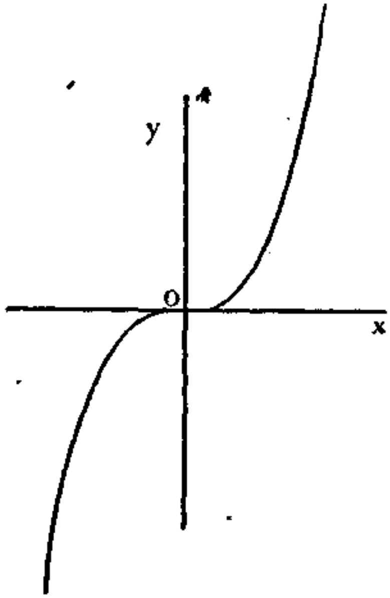

FIG. 95.

(2) The values of x^2 are *increasing before* the stationary point *and decreasing after.*

Such points on a curve are called turning-points.

It should be noted that not all stationary points are turning-points.

Consider the case of $y = x^3$ (Fig. 95), in which

$$\frac{dy}{dx} = 3x^2$$

We see that

$$3x^2 \text{ is } \textit{always positive;}$$

hence the function must be

always increasing.

Since, however,

$$\frac{dy}{dx} = 0 \text{ when } x = 0,$$

there is a *stationary point* at the origin, but the function is not increasing before this and decreasing afterwards, or vice versa. The slope θ remains as an acute angle throughout, except at the origin where θ is zero.

Hence the *stationary point is not a turning-point.*

It may be deduced also that if the differential coefficient of a function is always negative, the function itself is always decreasing, and there can be no turning-points: only stationary points are possible.

Examples

1. *For what value of x is there a turning-point on the curve of $y = 2x^2 - 6x + 9$, and what is the value of the function at the turning-point?*

If $$y = 2x^2 - 6x + 9$$

then $$\frac{dy}{dx} = 4x - 6$$

For a turning-point the differential coefficient must vanish.

$$\therefore \quad 4x - 6 = 0$$

and $$x = 1{\cdot}5$$

Thus there is one turning-point on the curve, where

$$x = 1{\cdot}5.$$

To find the value of the function at the turning-point substitute $x = 1{\cdot}5$ in the function.

Then
$$\begin{aligned} y &= 2(\tfrac{3}{2})^2 - 6(\tfrac{3}{2}) + 9 \\ &= 4{\cdot}5 - 9 + 9 \\ &= 4{\cdot}5. \end{aligned}$$

2. *For what values of x are there turning-points on the curve of $y = 4x^3 - x^2 - 2x + 1$?*

Now
$$\frac{dy}{dx} = 12x^2 - 2x - 2$$

For turning-points,
$$\frac{dy}{dx} = 0$$

$$\therefore \quad 12x^2 - 2x - 2 = 0$$
or
$$6x^2 - x - 1 = 0$$
$$\therefore \quad (3x + 1)(2x - 1) = 0$$
and
$$x = -\tfrac{1}{3} \text{ or } +\tfrac{1}{2}$$

$\therefore$ there are turning-points on the curve when
$$x = -\tfrac{1}{3}$$
and
$$x = +\tfrac{1}{2}$$

3. *For what values of x are there turning or stationary points on the curve of $y = 1 - x^3$?*

$$\frac{dy}{dx} = -3x^2$$

Now $-3x^2$ is negative for all real values of x.

$\therefore$ There are *no turning-points.*

But if
$$\frac{dy}{dx} = -3x^2 = 0$$
then
$$x = 0$$

$\therefore$ *there is a stationary point when $x = 0$.*

3. Maximum and Minimum Points

An examination of the curves of

$$y = x^2 \text{ and } y = -x^2 \text{ (Figs. 93 and 94)}$$

reveals a very important difference in the turning-points.

(1) In $y = x^2$, the turning-point is the lowest point on the curve. If other points are taken close to it and on either side of it, the value of the function at each of these is *greater* than at the turning-point. Such a value at a turning-point is called

a minimum value.

(2) In $y = -x^2$, examining the turning-point, we see that values just before and after it are *less* than at the turning-point itself. Such a value is called

a maximum value.

It is very important to be able to distinguish between maximum and minimum values, especially when a function has more than one of them. If the curve of the function has been drawn, it will generally be easy to see which is which, but an algebraical method must be found which will apply in all cases.

Let us first consider a function which has both maximum and minimum values.

Consider

$$y = (x - 1)(x - 2)(x - 3)$$

or when multiplied out,

$$y = x^3 - 6x^2 + 11x - 6.$$

It will be seen that the function will vanish when

$$x - 1 = 0, x - 2 = 0, x - 3 = 0$$

or when $\quad x = 1, 2 \text{ and } 3.$

Clearly if the curve cuts the axis at $x = 1$ and $x = 2$, and a very small increase in x produces a corresponding small increase in y, then there must be a turning-point between $x = 1$ and $x = 2$. Similarly there must be a turning-point between $x = 2$ and $x = 3$.

The shape of the curve is shown in Fig. 96. This shows that the turning-point between $x = 1$ and $x = 2$ is a maximum, and that between $x = 2$ and $x = 3$ a minimum.

Differentiating $y = x^3 - 6x^2 + 11x - 6$

$$\frac{dy}{dx} = 3x^2 - 12x + 11$$

For turning-points $3x^2 - 12x + 11 = 0$.
Solving, we find the roots of these equations are

$$x = 2{\cdot}58 \text{ or } 1{\cdot}42$$

$\therefore$ there are turning-points when $x = 2{\cdot}58$ and $x = 1{\cdot}42$.

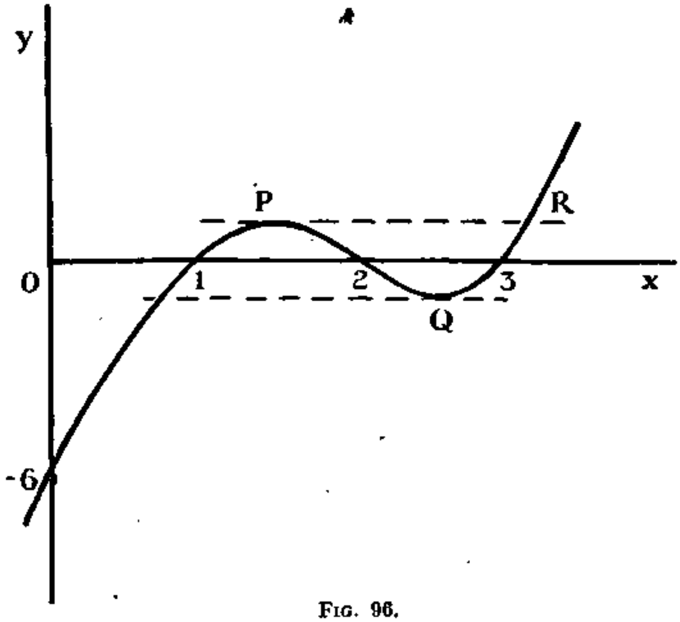

FIG. 96.

We must proceed to find an algebraical method of distinguishing which is a maximum and which a minimum.

It should be noted that the value at a maximum point such as P (Fig. 96) is not necessarily the greatest value of the function.

If PR be drawn from P parallel to the x axis, then all

points on the curve beyond R give values which are greater than that at P. Similarly there are values less than the minimum at Q.

4. To Distinguish between Maximum and Minimum Values

First Method.

We have seen that at a *maximum* point the value of the function is *greater* than at points very close to it, *i.e.*, for values of x a little greater and a little less. Similarly at a *minimum* point such values are *less*. We could therefore test whether the point is a maximum or minimum by substituting in the function values of x which are slightly greater or slightly less than the value at the turning-point, and then comparing the resulting values of the function with its value *at* the turning-point. This method is, however, usually somewhat tedious.

Second Method.

At a maximum point, since the function is increasing before it and decreasing afterwards,

$\therefore \frac{dy}{dx}$ must be positive before the point and negative afterwards. Similarly at a minimum point, $\frac{dy}{dx}$ must be negative before and positive afterwards. Thus if we substitute in the expression for $\frac{dy}{dx}$, values of x a little greater and a little less than at the turning-point, then

(a) *if* $\frac{dy}{dx}$ *is positive before and negative after, the point is a maximum;*

(b) *if* $\frac{dy}{dx}$ *is negative before and positive after, the point is a minimum.*

Example

Examine $y = 6x - x^2$ for maximum and minimum values.

If
$$y = 6x - x^2$$
$$\frac{dy}{dx} = 6 - 2x$$

$\therefore$ For turning-points, $6 - 2x = 0$
$$\therefore \quad x = 3$$

$\therefore$ There is one turning-point only—viz., when $x = 3$.

When $x = 2{\cdot}9$, $\frac{dy}{dx} = 6 - 2(2{\cdot}9)$ which is positive.

When $x = 3{\cdot}1$, $\frac{dy}{dx} = 6 - 2(3{\cdot}1)$ which is negative.

$\therefore$ When $x = 3$,

$6x - x^2$ has a maximum value.

Third Method.

We have seen that at a *maximum* point

(*a*) $\frac{dy}{dx}$ is zero;

(*b*) $\frac{dy}{dx}$ is positive before and negative afterwards.

$\therefore$ $\frac{dy}{dx}$ *must be decreasing.*

But if $\frac{dy}{dx}$ is decreasing, *its differential coefficient* must be *negative.*

$\therefore$ At a maximum point, $\frac{d^2y}{dx^2}$ is negative.

At a *minimum* point, similarly,

(*a*) $\frac{dy}{dx}$ is zero;

(*b*) $\frac{dy}{dx}$ is negative before and positive after.

$\therefore$ $\frac{dy}{dx}$ *is increasing*

$\therefore$ *its differential coefficient* must be *positive.*

$\therefore$ **At a minimum point $\frac{d^2y}{dx^2}$ is positive.**

These results may be summarised as follows:

	Maximum.	Minimum.
$y = f(x)$	increasing before. decreasing after.	decreasing before. increasing after.
$\frac{dy}{dx}$	positive before. negative after. $\therefore$ decreasing.	negative before. positive after. $\therefore$ increasing.
$\frac{d^2y}{dx^2}$	negative.	positive.

Examples

1. We will now apply these tests to the curve

$$y = (x - 1)(x - 2)(x - 3)$$

where we have previously seen there was a maximum point at $x = 1{\cdot}42$ and a minimum at $x = 2{\cdot}58$.

First Method.

$$\begin{aligned} \text{When } x = 1{\cdot}4,\quad y &= (1{\cdot}4 - 1)(1{\cdot}4 - 2)(1{\cdot}4 - 3) \\ &= (0{\cdot}4) \times (-0{\cdot}6) \times (-1{\cdot}6) \\ &= +0{\cdot}384 \end{aligned}$$

$$\begin{aligned} \text{When } x = 1{\cdot}42,\ y &= (1{\cdot}42 - 1)(1{\cdot}42 - 2)(1{\cdot}42 - 3) \\ &= (0{\cdot}42)(-0{\cdot}58)(-1{\cdot}58) \\ &= +0{\cdot}3849 \end{aligned}$$

$$\begin{aligned} \text{When } x = 1{\cdot}5,\quad y &= (1{\cdot}5 - 1)(1{\cdot}5 - 2)(1{\cdot}5 - 3) \\ &= (0{\cdot}5)(-0{\cdot}5)(-1{\cdot}5) \\ &= +0{\cdot}375 \end{aligned}$$

Thus in front of $x = 1{\cdot}42$ (from $x = 1{\cdot}4$ to $x = 1{\cdot}42$), y is increasing (from $0{\cdot}384$ to $0{\cdot}3849$),

while after $x = 1{\cdot}42$ (from $x = 1{\cdot}42$ to $x = 1{\cdot}5$),
y is decreasing (from $0{\cdot}3849$ to $0{\cdot}375$).

$\therefore$ *$x = 1{\cdot}42$ gives a maximum point.*

Again,

$$\begin{aligned}
\text{when}\quad x = 2{\cdot}5,\quad y &= (2{\cdot}5 - 1)(2{\cdot}5 - 2)(2{\cdot}5 - 3)\\
&= (1{\cdot}5)(0{\cdot}5)(-0{\cdot}5)\\
&= -0{\cdot}375\\
\text{when}\quad x = 2{\cdot}58,\quad y &= (2{\cdot}58 - 1)(2{\cdot}58 - 2)(2{\cdot}58 - 3)\\
&= (1{\cdot}58)(0{\cdot}58)(-0{\cdot}42)\\
&= -0{\cdot}3849\\
\text{when}\quad x = 3,\quad y &= (3 - 1)(3 - 2)(3 - 3)\\
&= 0
\end{aligned}$$

Thus in front of $x = 2{\cdot}58$ (from $x = 2{\cdot}5$ to $x = 2{\cdot}58$),
y is decreasing (from $-0{\cdot}375$ to $-0{\cdot}3849$),
while after $x = 2{\cdot}58$ (from $x = 2{\cdot}58$ to $x = 3$),
y is increasing (from $-0{\cdot}3849$ to 0).

$\therefore$ *$x = 2{\cdot}58$ gives a minimum point.*

Second Method.

Since $$y = (x - 1)(x - 2)(x - 3)$$
or $$y = x^3 - 6x^2 + 11x - 6$$
$$\therefore \quad \frac{dy}{dx} = 3x^2 - 12x + 11.$$

$$\begin{aligned}
\text{When } x = 1{\cdot}4,\ \frac{dy}{dx} &= 3(1{\cdot}4)^2 - 12(1{\cdot}4) + 11\\
&= 5{\cdot}88 - 16{\cdot}8 + 11\\
&= 16{\cdot}88 - 16{\cdot}8\\
&= 0{\cdot}08 \text{ which is } \textit{positive}
\end{aligned}$$

$$\begin{aligned}
\text{When } x = 1{\cdot}5,\ \frac{dy}{dx} &= 3(1{\cdot}5)^2 - 12(1{\cdot}5) + 11\\
&= 6{\cdot}75 - 18 + 11\\
&= 17{\cdot}75 - 18\\
&= -0{\cdot}25, \text{ which is } \textit{negative}
\end{aligned}$$

$\therefore \quad \dfrac{dy}{dx}$ is positive before $x = 1{\cdot}42$ and negative after.

$\therefore \quad$ *$x = 1{\cdot}42$ gives a maximum point.*

$$\text{When } x = 2{\cdot}5,\ \frac{dy}{dx} = 3(2{\cdot}5)^2 - 12(2{\cdot}5) + 11$$
$$= 18{\cdot}75 - 30 + 11$$
$$= 29{\cdot}75 - 30$$
$$= -0{\cdot}25 \text{ which is } \textit{negative.}$$

$$\text{When } x = 3,\quad \frac{dy}{dx} = 3(3^2) - 12(3) + 11$$
$$= 27 - 36 + 11$$
$$= 38 - 36$$
$$= 2 \text{ which is } \textit{positive.}$$

$\therefore \quad \dfrac{dy}{dx}$ is negative before $x = 2{\cdot}58$ and positive after.

$\therefore \quad$ *$x = 2{\cdot}58$ gives a minimum point.*

Third Method.

$$\text{Since} \qquad \frac{dy}{dx} = 3x^2 - 12x + 11$$
$$\frac{d^2y}{dx^2} = 6x - 12$$
$$= 6(x - 2)$$

$$\text{When} \qquad x = 1{\cdot}42,\ \frac{d^2y}{dx^2} = 6(1{\cdot}42 - 2)$$
$$= (6)(-0{\cdot}58)$$
$$= \text{a } \textit{negative} \text{ value.}$$

$$\text{When} \qquad x = 2{\cdot}58,\ \frac{d^2y}{dx^2} = 6(2{\cdot}58 - 2)$$
$$= (6)(0{\cdot}58)$$
$$= \text{a } \textit{positive} \text{ value.}$$

Thus

$x = 1{\cdot}42$ gives a maximum point
$x = 2{\cdot}58$ gives a minimum point.

The student thus has the choice of three methods, which have been shown to lead to the same conclusions; but the third method (of using the second differential coefficient) is usually to be preferred owing to its simplicity.

2. *Examine $y = 3x - x^3$ for maximum and minimum values.*

$$y = 3x - x^3$$

$$\therefore \quad \frac{dy}{dx} = 3 - 3x^2$$

For turning-points, $\quad \frac{dy}{dx} = 3 - 3x^2 = 0$

$$\therefore \quad 3(1 - x^2) = 0$$

$$\therefore \quad x = \pm 1.$$

But $\quad \frac{d^2y}{dx^2} = -6x$

When $\quad x = +1, \frac{d^2y}{dx^2} = (-6)(+1)$

$$= -6 \text{ (i.e., \textit{negative})}$$

When $\quad x = -1, \frac{d^2y}{dx^2} = (-6)(-1)$

$$= +6 \text{ (i.e., \textit{positive})}$$

$\therefore \quad 3x - x^3$ has

a maximum value when $x = +1$
a minimum value when $x = -1$.

3. *Examine $x^3 - 6x^2 - 15x + 10$' for maximum and minimum values. Also find the maximum and minimum values of the function.*

$$y = x^3 - 6x^2 - 15x + 10$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 12x - 15$$

$$= 3(x^2 - 4x - 5)$$

$\therefore$ For turning-points

$$x^2 - 4x - 5 = 0$$

$$\therefore \quad x = 5 \text{ or } -1.$$

But $$\frac{d^2y}{dx^2} = 6x - 12$$
$$= 6(x - 2)$$

When $$x = 5, \frac{d^2y}{dx^2} = 6(5 - 2) = +18$$

When $$x = -1, \frac{d^2y}{dx^2} = 6(-1 - 2) = -18.$$

$\therefore$ $x^3 - 6x^2 - 15x + 10$ has
a maximum value when $x = -1$,
a minimum value when $x = 5$.

The actual maximum and minimum values are found by substituting $x = -1$ and $+5$, respectively, in

$$x^3 - 6x^2 - 15x + 10$$

When $x = -1$, $x^3 - 6x^2 - 15x + 10 = +18$
When $x = 5$, $x^3 - 6x^2 - 15x + 10 = -90.$

Thus

the maximum value of $x^3 - 6x^2 - 15x + 10$ is 18 and occurs when $x = -1$,
the minimum value of $x^3 - 6x^2 - 15x + 10$ is -90 and occurs when $x = 5$.

Practical Applications

1. *An open tank is to be made of sheet iron; it must have a square base and vertical sides, with a capacity of 8 cub. ft. Find its width and depth so as to use the least amount of sheet iron.*

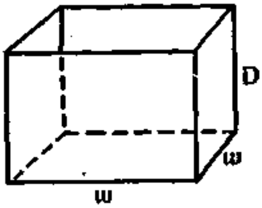

FIG. 97.

Let its width and length $= w$ ft. (Fig. 97)
Let its depth $= \text{D}$ ft.
$\therefore$ Volume V $= \text{D}w^2$ cub. ft.
and area A of sheet iron $= 4\text{D}w + w^2$ sq. ft.

$\therefore$ $\text{A} = 4\text{D}w + w^2$ must be a minimum.
$\therefore$ The differential coefficient of $4\text{D}w + w^2$ must be zero.

As $4Dw + w^2$ contains two variables D and w, we cannot differentiate at this stage.

But $$V = Dw^2 = 8$$

$$\therefore \quad D = \frac{8}{w^2}$$

Substituting this value in $4Dw + w^2$,

$$A = 4 \cdot \frac{8}{w^2} \cdot w + w^2$$
$$= \frac{32}{w} + w^2$$
$$= 32w^{-1} + w^2$$
$$\therefore \quad \frac{dA}{dw} = -32w^{-2} + 2w$$
$$= -\frac{32}{w^2} + 2w$$

$\therefore$ For a maximum or minimum value of A,

$$-\frac{32}{w^2} + 2w = 0$$
$$\therefore \quad 2w^3 = 32$$
$$\therefore \quad w = \sqrt[3]{16}$$
$$\therefore \quad w = 2{\cdot}52$$

But $$\frac{d^2A}{dw^2} = +\frac{64}{w^3} + 2$$

and when $$w = 2{\cdot}52$$

$$\frac{d^2A}{dw^2} = +\frac{64}{(2{\cdot}52)^3} + 2, \text{ which is positive}$$

$\therefore \quad w = 2{\cdot}52$ gives a minimum value to A.

Also, since $$D = \frac{8}{w^2}$$

$$D = \frac{8}{(2{\cdot}52)^2} = 1{\cdot}26$$

$\therefore$ For a minimum amount of sheet iron to be used
the width and length $=$ *2·52 ft.*
and the depth $=$ *1·26 ft.*

2. *A rectangular sheet of tin 30 × 24 ins. has four equal squares cut out at the corners, and the sides are then turned up to form a rectangular box. What must be the size of the squares cut away so that the volume of the box may be as great as possible?*

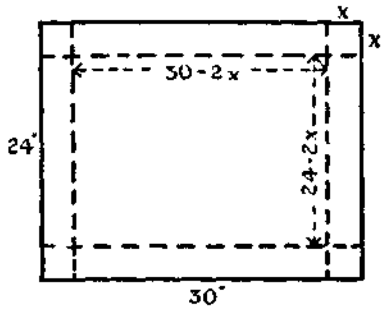

FIG. 98.

Let the side of each square $= x$ ins. (Fig. 98).
After the box is formed

its length $= (30 - 2x)$ ins.
its breadth $= (24 - 2x)$ ins.
its depth $= x$ ins.

$\therefore$ Volume of box $= (30 - 2x)(24 - 2x)(x)$ cub. ins.

$\therefore$ $V = 720x - 108x^2 + 4x^3$ cub. ins.

Thus, for V to be a maximum (or a minimum),

$$\frac{dV}{dx} = 720 - 216x + 12x^2 = 0.$$

$$\therefore \quad x^2 - 18x + 60 = 0$$

$$\therefore \quad x = 13{\cdot}583 \text{ or } 4{\cdot}417 \text{ ins.}$$

But $$\frac{d^2V}{dx^2} = -216 + 24x$$

$$= 24(x - 9)$$

When $x = 13{\cdot}583, \dfrac{d^2V}{dx^2} = 24(13{\cdot}583 - 9)$ which is positive;

When $x = 4{\cdot}417, \dfrac{d^2V}{dx^2} = 24(4{\cdot}417 - 9)$ which is negative.

$\therefore$ V, the volume of the box, is as great as possible *when the sides of the squares cut out are 4·417 ins.*

NOTE—It should be noted that 13·583 is an impossible value in this problem.

3. *The cost, £C, per mile of an electric cable is given by*

$$C = \frac{120}{x} + 600x$$

where x is its cross-section in sq. ins. Find the cross-section for which the cost is least, and also the least cost per mile.

$$C = \frac{120}{x} + 600x$$

$$\therefore \quad \frac{dC}{dx} = -\frac{120}{x^2} + 600$$

$\therefore$ C has a maximum or a minimum value when

$$\frac{120}{x^2} = 600$$

$$\therefore \quad x = \sqrt{0{\cdot}2}$$

$$\therefore \quad x = 0{\cdot}447 \text{ ins.},$$

the negative value being inadmissible

$\dfrac{d^2C}{dx^2} = \dfrac{240}{x^3}$, which is positive when $x = 0{\cdot}447$

$\therefore$ When $x = 0{\cdot}447$ in., the cost is a minimum.

The actual minimum cost

$$= \frac{120}{x} + 600x,$$

where $x = 0{\cdot}447$

$\therefore$ $\left.\begin{array}{c}\text{Minimum cost}\\ \text{per mile}\end{array}\right\} = $ *£536 12s.*

EXERCISE 20

Find the turning-point on each of the following, and state whether the function has a maximum or a minimum value in each case :

1. $3x^2 - 2x$.	3. $10x - 2x^2$.	5. $24x - x^2$.
2. $x^2 + 3x$.	4. $2x^2 + x$.	6. $12x + 4x^2$.

Find the turning-points of each and also the maximum and minimum values.

7. $3x^3 - 9x^2 - 27x + 10$.
8. $x^3 - 9x^2 + 15x - 7$.
9. $24x + 5x^2 - 2x^3$.
10. [illegible] $- 21x^2 + 36x + 3$.
11. [illegible] $- 9x + 6x^2 - x^3$.
12. Divide 50 into 2 parts such that their product is a maximum.
13. Find a number which exceeds its square by the greatest possible amount.
14. What number added to its reciprocal gives a minimum sum?
15. A closed cylindrical tin is required to contain 200 cub. ins. Find the ratio of the height to the diameter so as to use the minimum amount of metal.
16. If the strength of a rectangular beam of wood varies as its breadth and the square of its depth, find the dimensions of the strongest beam that can be cut out of a round log of diameter d.
17. It is required to enclose 2 acres of land in the form of a rectangle. Find its length and breadth so as to need the shortest possible fence.
18. The relation between C, the cost per hour, of running a certain ship, and its speed, v knots, is given by

$$C = 1{\cdot}25 + 0{\cdot}0008v^3.$$

Find the average speed which is most economical for a voyage of 800 miles.

19. When a belt runs at v ft. per sec. the horse-power transmitted is given by

$$\text{H.P.} = \frac{v}{1000}\left\{T - \frac{mv^2}{g}\right\}$$

where T is the maximum safe tension allowed in the belt, m is the mass of the belt per foot length, $g = 32$.

Find the speed at which the maximum H.P. is transmitted if $T = 400$ lb. per sq. in., and $m = 0{\cdot}5$ lb.

Also find the maximum H.P.

20. In a certain type of engine the ratio of expansion, r, and the number of lb., N, of steam used per I.H.P. hour are related by :

$$N = 0{\cdot}52r^2 - 5{\cdot}5r + 32.$$

Find the value of r which gives the minimum value of N (*i.e.*, the most economic ratio of expansion). Find also the corresponding value of N.

21. Find the maximum value of the function

$$1 + 4x - 2x^2. \qquad \text{(U.E.I.)}$$

22. An engineer has sufficient expanded metal 6 ft. high to build a screen 30 ft. in length. He intends using this to fence in a rectangular piece of his workshop as a tool store, using the wall as one side. Use the differential calculus to calculate the dimensions of the largest store he can arrange. (U.E.I.)

CONSTANTS

Constant.	Number.	Log.
π	3·1416	0·49715
$\frac{\pi}{4}$	0·7854	$\bar{1}$·89509
$\frac{1}{\pi}$	0·3183	$\bar{1}$·50285
π^2	9·8696	0·99430
$\sqrt{\pi}$	1·7725	0·24857
$\frac{4}{3}\pi$	4·1888	0·62209
$\frac{180}{\pi}$	57·2958	1·75812
$\frac{\pi}{180}$	0·01745	$\bar{2}$·24188
e	2·71828	0·43429
$\text{Log}_e\ 10$	2·3026	0·36222

CONVERSION FACTORS

To convert	Multiply by	Log.
Metres to inches	39·37	1·59517
Inches to centimetres	2·5400	0·40483
Kilometres to miles	0·6214	$\bar{1}$·79335
Kilograms to lb.	2·20462	0·34333
Lb. to kilograms	0·45359	$\bar{1}$·65666
Gallons to cubic inches	277·45	2·44318
Radians to degrees	57·2958	1·75812
Miles per hour to feet per second	1·4666	0·1663

G. (at Greenwich) = 32·191 ft. per sec.^{2}
= 981·18 cm. per sec.^{2}

Weight of 1 cub. ft. of water = 62·42 lb. (at 4° C.)

No.	Log.	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1·0	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1·1	·0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
1·2	·0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1·3	·1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1·4	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1·5	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1·6	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1·7	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1·8	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1·9	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2·0	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2·1	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2·2	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2·3	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
2·4	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
2·5	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2·6	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
2·7	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
2·8	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
2·9	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
3·0	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
3·1	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
3·2	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
3·3	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
3·4	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
3·5	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
3·6	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
3·7	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
3·8	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
3·9	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
4·0	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
4·1	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
4·2	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
4·3	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
4·4	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
4·5	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
4·6	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
4·7	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
4·8	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
4·9	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
5·0	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
5·1	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
5·2	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
5·3	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
5·4	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

No.	Log.	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5·5	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
5·6	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
5·7	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
5·8	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
5·9	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
6·0	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
6·1	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
6·2	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
6·3	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
6·4	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
6·5	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
6·6	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
6·7	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
6·8	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
6·9	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
7·0	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
7·1	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
7·2	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
7·3	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
7·4	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
7·5	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
7·6	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
7·7	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
7·8	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
7·9	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
8·0	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
8·1	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
8·2	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
8·3	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
8·4	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
8·5	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
8·6	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
8·7	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
8·8	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
8·9	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
9·0	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
9·1	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
9·2	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
9·3	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
9·4	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
9·5	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
9·6	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
9·7	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
9·8	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
9·9	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTI-LOGARITHMS.

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTI-LOGARITHMS.

Log.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES.

Angle	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	3	6	9	12	15
1°	·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	3	6	9	12	15
2°	·0349	·0366	·0384	·0401	·0419	·0436	·0454	·0471	·0488	·0506	3	6	9	12	15
3°	·0523	·0541	·0558	·0576	·0593	·0610	·0628	·0645	·0663	·0680	3	6	9	12	15
4°	·0698	·0715	·0732	·0750	·0767	·0785	·0802	·0819	·0837	·0854	3	6	9	12	14
5°	·0872	·0889	·0906	·0924	·0941	·0958	·0976	·0993	·1011	·1028	3	6	9	12	14
6°	·1045	·1063	·1080	·1097	·1115	·1132	·1149	·1167	·1184	·1201	3	6	9	12	14
7°	·1219	·1236	·1253	·1271	·1288	·1305	·1323	·1340	·1357	·1374	3	6	9	12	14
8°	·1392	·1409	·1426	·1444	·1461	·1478	·1495	·1513	·1530	·1547	3	6	9	12	14
9°	·1564	·1582	·1599	·1616	·1633	·1650	·1668	·1685	·1702	·1719	3	6	9	12	14
10°	·1736	·1754	·1771	·1788	·1805	·1822	·1840	·1857	·1874	·1891	3	6	9	11	14
11°	·1908	·1925	·1942	·1959	·1977	·1994	·2011	·2028	·2045	·2062	3	6	9	11	14
12°	·2079	·2096	·2113	·2130	·2147	·2164	·2181	·2198	·2215	·2233	3	6	9	11	14
13°	·2250	·2267	·2284	·2300	·2317	·2334	·2351	·2368	·2385	·2402	3	6	8	11	14
14°	·2419	·2436	·2453	·2470	·2487	·2504	·2521	·2538	·2554	·2571	3	6	8	11	14
15°	·2588	·2605	·2622	·2639	·2656	·2672	·2689	·2706	·2723	·2740	3	6	8	11	14
16°	·2756	·2773	·2790	·2807	·2823	·2840	·2857	·2874	·2890	·2907	3	6	8	11	14
17°	·2924	·2940	·2957	·2974	·2990	·3007	·3024	·3040	·3057	·3074	3	6	8	11	14
18°	·3090	·3107	·3123	·3140	·3156	·3173	·3190	·3206	·3223	·3239	3	6	8	11	14
19°	·3256	·3272	·3289	·3305	·3322	·3338	·3355	·3371	·3387	·3404	3	5	8	11	14
20°	·3420	·3437	·3453	·3469	·3486	·3502	·3518	·3535	·3551	·3567	3	5	8	11	14
21°	·3584	·3600	·3616	·3633	·3649	·3665	·3681	·3697	·3714	·3730	3	5	8	11	14
22°	·3746	·3762	·3778	·3795	·3811	·3827	·3843	·3859	·3875	·3891	3	5	8	11	14
23°	·3907	·3923	·3939	·3955	·3971	·3987	·4003	·4019	·4035	·4051	3	5	8	11	14
24°	·4067	·4083	·4099	·4115	·4131	·4147	·4163	·4179	·4195	·4210	3	5	8	11	13
25°	·4226	·4242	·4258	·4274	·4289	·4305	·4321	·4337	·4352	·4368	3	5	8	11	13
26°	·4384	·4399	·4415	·4431	·4446	·4462	·4478	·4493	·4509	·4524	3	5	8	10	13
27°	·4540	·4555	·4571	·4586	·4602	·4617	·4633	·4648	·4664	·4679	3	5	8	10	13
28°	·4695	·4710	·4726	·4741	·4756	·4772	·4787	·4802	·4818	·4833	3	5	8	10	13
29°	·4848	·4863	·4879	·4894	·4909	·4924	·4939	·4955	·4970	·4985	3	5	8	10	13
30°	·5000	·5015	·5030	·5045	·5060	·5075	·5090	·5105	·5120	·5135	3	5	8	10	13
31°	·5150	·5165	·5180	·5195	·5210	·5225	·5240	·5255	·5270	·5284	2	5	7	10	12
32°	·5299	·5314	·5329	·5344	·5358	·5373	·5388	·5402	·5417	·5432	2	5	7	10	12
33°	·5446	·5461	·5476	·5490	·5505	·5519	·5534	·5548	·5563	·5577	2	5	7	10	12
34°	·5592	·5606	·5621	·5635	·5650	·5664	·5678	·5693	·5707	·5721	2	5	7	10	12
35°	·5736	·5750	·5764	·5779	·5793	·5807	·5821	·5835	·5850	·5864	2	5	7	9	12
36°	·5878	·5892	·5906	·5920	·5934	·5948	·5962	·5976	·5990	·6004	2	5	7	9	12
37°	·6018	·6032	·6046	·6060	·6074	·6088	·6101	·6115	·6129	·6143	2	5	7	9	12
38°	·6157	·6170	·6184	·6198	·6211	·6225	·6239	·6252	·6266	·6280	2	5	7	9	11
39°	·6293	·6307	·6320	·6334	·6347	·6361	·6374	·6388	·6401	·6414	2	4	7	9	11
40°	·6428	·6441	·6455	·6468	·6481	·6494	·6508	·6521	·6534	·6547	2	4	7	9	11
41°	·6561	·6574	·6587	·6600	·6613	·6626	·6639	·6652	·6665	·6678	2	4	7	9	11
42°	·6691	·6704	·6717	·6730	·6743	·6756	·6769	·6782	·6794	·6807	2	4	6	9	11
43°	·6820	·6833	·6845	·6858	·6871	·6884	·6896	·6909	·6921	·6934	2	4	6	8	11
44°	·6947	·6959	·6972	·6984	·6997	·7009	·7022	·7034	·7046	·7059	2	4	6	8	10

NATURAL SINES.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4	5′
45°	·7071	·7083	·7096	·7108	·7120	·7133	·7145	·7157	·7169	·7181	2	4	6	8	10
46°	·7193	·7206	·7218	·7230	·7242	·7254	·7266	·7278	·7290	·7302	2	4	6	8	10
47°	·7314	·7325	·7337	·7349	·7361	·7373	·7385	·7396	·7408	·7420	2	4	6	8	10
48°	·7431	·7443	·7455	·7466	·7478	·7490	·7501	·7513	·7524	·7536	2	4	6	8	10
49°	·7547	·7559	·7570	·7581	·7593	·7604	·7615	·7627	·7638	·7649	2	4	6	8	9
50°	·7660	·7672	·7683	·7694	·7705	·7716	·7727	·7738	·7749	·7760	2	4	6	7	9
51°	·7771	·7782	·7793	·7804	·7815	·7826	·7837	·7848	·7859	·7869	2	4	5	7	9
52°	·7880	·7891	·7902	·7912	·7923	·7934	·7944	·7955	·7965	·7976	2	4	5	7	9
53°	·7986	·7997	·8007	·8018	·8028	·8039	·8049	·8059	·8070	·8080	2	3	5	7	9
54°	·8090	·8100	·8111	·8121	·8131	·8141	·8151	·8161	·8171	·8181	2	3	5	7	8
55°	·8192	·8202	·8211	·8221	·8231	·8241	·8251	·8261	·8271	·8281	2	3	5	7	8
56°	·8290	·8300	·8310	·8320	·8329	·8339	·8348	·8358	·8368	·8377	2	3	5	6	8
57°	·8387	·8396	·8406	·8415	·8425	·8434	·8443	·8453	·8462	·8471	2	3	5	6	8
58°	·8480	·8490	·8499	·8508	·8517	·8526	·8536	·8545	·8554	·8563	2	3	5	6	8
59°	·8572	·8581	·8590	·8599	·8607	·8616	·8625	·8634	·8643	·8652	1	3	4	6	7
60°	·8660	·8669	·8678	·8686	·8695	·8704	·8712	·8721	·8729	·8738	1	3	4	6	7
61°	·8746	·8755	·8763	·8771	·8780	·8788	·8796	·8805	·8813	·8821	1	3	4	6	7
62°	·8829	·8838	·8846	·8854	·8862	·8870	·8878	·8886	·8894	·8902	1	3	4	5	7
63°	·8910	·8918	·8926	·8934	·8942	·8949	·8957	·8965	·8973	·8980	1	3	4	5	6
64°	·8988	·8996	·9003	·9011	·9018	·9026	·9033	·9041	·9048	·9056	1	3	4	5	6
65°	·9063	·9070	·9078	·9085	·9092	·9100	·9107	·9114	·9121	·9128	1	2	4	5	6
66°	·9135	·9143	·9150	·9157	·9164	·9171	·9178	·9184	·9191	·9198	1	2	3	5	6
67°	·9205	·9212	·9219	·9225	·9232	·9239	·9245	·9252	·9259	·9265	1	2	3	4	6
68°	·9272	·9278	·9285	·9291	·9298	·9304	·9311	·9317	·9323	·9330	1	2	3	4	5
69°	·9336	·9342	·9348	·9354	·9361	·9367	·9373	·9379	·9385	·9391	1	2	3	4	5
70°	·9397	·9403	·9409	·9415	·9421	·9426	·9432	·9438	·9444	·9449	1	2	3	4	5
71°	·9455	·9461	·9466	·9472	·9478	·9483	·9489	·9494	·9500	·9505	1	2	3	4	5
72°	·9511	·9516	·9521	·9527	·9532	·9537	·9542	·9548	·9553	·9558	1	2	3	3	4
73°	·9563	·9568	·9573	·9578	·9583	·9588	·9593	·9598	·9603	·9608	1	2	2	3	4
74°	·9613	·9617	·9622	·9627	·9632	·9636	·9641	·9646	·9650	·9655	1	2	2	3	4
75°	·9659	·9664	·9668	·9673	·9677	·9681	·9686	·9690	·9694	·9699	1	1	2	3	4
76°	·9703	·9707	·9711	·9715	·9720	·9724	·9728	·9732	·9736	·9740	1	1	2	3	3
77°	·9744	·9748	·9751	·9755	·9759	·9763	·9767	·9770	·9774	·9778	1	1	2	3	3
78°	·9781	·9785	·9789	·9792	·9796	·9799	·9803	·9806	·9810	·9813	1	1	2	2	3
79°	·9816	·9820	·9823	·9826	·9829	·9833	·9836	·9839	·9842	·9845	1	1	2	2	3
80°	·9848	·9851	·9854	·9857	·9860	·9863	·9866	·9869	·9871	·9874	0	1	1	2	2
81°	·9877	·9880	·9882	·9885	·9888	·9890	·9893	·9895	·9898	·9900	0	1	1	2	2
82°	·9903	·9905	·9907	·9910	·9912	·9914	·9917	·9919	·9921	·9923	0	1	1	2	2
83°	·9925	·9928	·9930	·9932	·9934	·9936	·9938	·9940	·9942	·9943	0	1	1	1	2
84°	·9945	·9947	·9949	·9951	·9952	·9954	·9956	·9957	·9959	·9960	0	1	1	1	1
85°	·9962	·9963	·9965	·9966	·9968	·9969	·9971	·9972	·9973	·9974	0	0	1	1	1
86°	·9976	·9977	·9978	·9979	·9980	·9981	·9982	·9983	·9984	·9985	0	0	1	1	1
87°	·9986	·9987	·9988	·9989	·9990	·9990	·9991	·9992	·9993	·9993	0	0	0	1	1
88°	·9994	·9995	·9995	·9996	·9996	·9997	·9997	·9997	·9998	·9998	0	0	0	1	0
89°	·9998	·9999	·9999	·9999	·9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0

NATURAL COSINES.

Subtract Differences

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	1·000	1·000	1·000	1·000	1·000	·9999	·9999	·9999	·9999	0	0	0	0	0
1°	·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995	·9995	0	0	0	0	0
2°	·9994	·9993	·9993	·9992	·9991	·9990	·9990	·9989	·9988	·9987	0	0	0	0	0
3°	·9986	·9985	·9984	·9983	·9982	·9981	·9980	·9979	·9978	·9977	0	0	1	1	1
4°	·9976	·9974	·9973	·9972	·9971	·9969	·9968	·9966	·9965	·9963	0	0	1	1	1
5°	·9962	·9960	·9959	·9957	·9956	·9954	·9952	·9951	·9949	·9947	0	1	1	1	1
6°	·9945	·9943	·9942	·9940	·9938	·9936	·9934	·9932	·9930	·9928	0	1	1	1	2
7°	·9925	·9923	·9921	·9919	·9917	·9914	·9912	·9910	·9907	·9905	0	1	1	2	2
8°	·9903	·9900	·9898	·9895	·9893	·9890	·9888	·9885	·9882	·9880	0	1	1	2	2
9°	·9877	·9874	·9871	·9869	·9866	·9863	·9860	·9857	·9854	·9851	0	1	1	2	2
10°	·9848	·9845	·9842	·9839	·9836	·9833	·9829	·9826	·9823	·9820	1	1	2	2	3
11°	·9816	·9813	·9810	·9806	·9803	·9799	·9796	·9792	·9789	·9785	1	1	2	2	3
12°	·9781	·9778	·9774	·9770	·9767	·9763	·9759	·9755	·9751	·9748	1	1	2	3	3
13°	·9744	·9740	·9736	·9732	·9728	·9724	·9720	·9715	·9711	·9707	1	1	2	3	3
14°	·9703	·9699	·9694	·9690	·9686	·9681	·9677	·9673	·9668	·9664	1	1	2	3	4
15°	·9659	·9655	·9650	·9646	·9641	·9636	·9632	·9627	·9622	·9617	1	2	2	3	4
16°	·9613	·9608	·9603	·9598	·9593	·9588	·9583	·9578	·9573	·9568	1	2	2	3	4
17°	·9563	·9558	·9553	·9548	·9542	·9537	·9532	·9527	·9521	·9516	1	2	3	4	4
18°	·9511	·9505	·9500	·9494	·9489	·9483	·9478	·9472	·9466	·9461	1	2	3	4	5
19°	·9455	·9449	·9444	·9438	·9432	·9426	·9421	·9415	·9409	·9403	1	2	3	4	5
20°	·9397	·9391	·9385	·9379	·9373	·9367	·9361	·9354	·9348	·9342	1	2	3	4	5
21°	·9336	·9330	·9323	·9317	·9311	·9304	·9298	·9291	·9285	·9278	1	2	3	4	5
22°	·9272	·9265	·9259	·9252	·9245	·9239	·9232	·9225	·9219	·9212	1	2	3	4	6
23°	·9205	·9198	·9191	·9184	·9178	·9171	·9164	·9157	·9150	·9143	1	2	3	5	6
24°	·9135	·9128	·9121	·9114	·9107	·9100	·9092	·9085	·9078	·9070	1	2	4	5	6
25°	·9063	·9056	·9048	·9041	·9033	·9026	·9018	·9011	·9003	·8996	1	3	4	5	6
26°	·8988	·8980	·8973	·8965	·8957	·8949	·8942	·8934	·8926	·8918	1	3	4	5	6
27°	·8910	·8902	·8894	·8886	·8878	·8870	·8862	·8854	·8846	·8838	1	3	4	5	7
28°	·8829	·8821	·8813	·8805	·8796	·8788	·8780	·8771	·8763	·8755	1	3	4	6	7
29°	·8746	·8738	·8729	·8721	·8712	·8704	·8695	·8686	·8678	·8669	1	3	4	6	7
30°	·8660	·8652	·8643	·8634	·8625	·8616	·8607	·8599	·8590	·8581	1	3	4	6	7
31°	·8572	·8563	·8554	·8545	·8536	·8526	·8517	·8508	·8499	·8490	2	3	5	6	8
32°	·8480	·8471	·8462	·8453	·8443	·8434	·8425	·8415	·8406	·8396	2	3	5	6	8
33°	·8387	·8377	·8368	·8358	·8348	·8339	·8329	·8320	·8310	·8300	2	3	5	6	8
34°	·8290	·8281	·8271	·8261	·8251	·8241	·8231	·8221	·8211	·8202	2	3	5	7	8
35°	·8192	·8181	·8171	·8161	·8151	·8141	·8131	·8121	·8111	·8100	2	3	5	7	8
36°	·8090	·8080	·8070	·8059	·8049	·8039	·8028	·8018	·8007	·7997	2	3	5	7	9
37°	·7986	·7976	·7965	·7955	·7944	·7934	·7923	·7912	·7902	·7891	2	4	5	7	9
38°	·7880	·7869	·7859	·7848	·7837	·7826	·7815	·7804	·7793	·7782	2	4	5	7	9
39°	·7771	·7760	·7749	·7738	·7727	·7716	·7705	·7694	·7683	·7672	2	4	6	7	9
40°	·7660	·7649	·7638	·7627	·7615	·7604	·7593	·7581	·7570	·7559	2	4	6	8	9
41°	·7547	·7536	·7524	·7513	·7501	·7490	·7478	·7466	·7455	·7443	2	4	6	8	10
42°	·7431	·7420	·7408	·7396	·7385	·7373	·7361	·7349	·7337	·7325	2	4	6	8	10
43°	·7314	·7302	·7290	·7278	·7266	·7254	·7242	·7230	·7218	·7206	2	4	6	8	10
44°	·7193	·7181	·7169	·7157	·7145	·7133	·7120	·7108	·7096	·7083	2	4	6	8	10

NATURAL COSINES.

Subtract Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	·7059	·7046	·7034	·7022	·7009	·6997	·6984	·6972	·6959	2	4	6	8	10
46°	·6947	·6934	·6921	·6909	·6896	·6884	·6871	·6858	·6845	·6833	2	4	6	8	11
47°	·6820	·6807	·6794	·6782	·6769	·6756	·6743	·6730	·6717	·6704	2	4	6	9	11
48°	·6691	·6678	·6665	·6652	·6639	·6626	·6613	·6600	·6587	·6574	2	4	7	9	11
49°	·6561	·6547	·6534	·6521	·6508	·6494	·6481	·6468	·6455	·6441	2	4	7	9	11
50°	·6428	·6414	·6401	·6388	·6374	·6361	·6347	·6334	·6320	·6307	2	4	7	9	11
51°	·6293	·6280	·6266	·6252	·6239	·6225	·6211	·6198	·6184	·6170	2	5	7	9	11
52°	·6157	·6143	·6129	·6115	·6101	·6088	·6074	·6060	·6046	·6032	2	5	7	9	12
53°	·6018	·6004	·5990	·5976	·5962	·5948	·5934	·5920	·5906	·5892	2	5	7	9	12
54°	·5878	·5864	·5850	·5835	·5821	·5807	·5793	·5779	·5764	·5750	2	5	7	9	12
55°	·5736	·5721	·5707	·5693	·5678	·5664	·5650	·5635	·5621	·5606	2	5	7	10	12
56°	·5592	·5577	·5563	·5548	·5534	·5519	·5505	·5490	·5476	·5461	2	5	7	10	12
57°	·5446	·5432	·5417	·5402	·5388	·5373	·5358	·5344	·5329	·5314	2	5	7	10	12
58°	·5299	·5284	·5270	·5255	·5240	·5225	·5210	·5195	·5180	·5165	2	5	7	10	12
59°	·5150	·5135	·5120	·5105	·5090	·5075	·5060	·5045	·5030	·5015	3	5	8	10	13
60°	·5000	·4985	·4970	·4955	·4939	·4924	·4909	·4894	·4879	·4863	3	5	8	10	13
61°	·4848	·4833	·4818	·4802	·4787	·4772	·4756	·4741	·4726	·4710	3	5	8	10	13
62°	·4695	·4679	·4664	·4648	·4633	·4617	·4602	·4586	·4571	·4555	3	5	8	10	13
63°	·4540	·4524	·4509	·4493	·4478	·4462	·4446	·4431	·4415	·4399	3	5	8	10	13
64°	·4384	·4368	·4352	·4337	·4321	·4305	·4289	·4274	·4258	·4242	3	5	8	11	13
65°	·4226	·4210	·4195	·4179	·4163	·4147	·4131	·4115	·4099	·4083	3	5	8	11	13
66°	·4067	·4051	·4035	·4019	·4003	·3987	·3971	·3955	·3939	·3923	3	5	8	11	13
67°	·3907	·3891	·3875	·3859	·3843	·3827	·3811	·3795	·3778	·3762	3	5	8	11	13
68°	·3746	·3730	·3714	·3697	·3681	·3665	·3649	·3633	·3616	·3600	3	5	8	11	14
69°	·3584	·3567	·3551	·3535	·3518	·3502	·3486	·3469	·3453	·3437	3	5	8	11	14
70°	·3420	·3404	·3387	·3371	·3355	·3338	·3322	·3305	·3289	·3272	3	5	8	11	14
71°	·3256	·3239	·3223	·3206	·3190	·3173	·3156	·3140	·3123	·3107	3	6	8	11	14
72°	·3090	·3074	·3057	3040	·3024	·3007	·2990	·2974	·2957	·2940	3	6	8	11	14
73°	·2924	·2907	·2890	·2874	·2857	·2840	·2823	·2807	·2790	·2773	3	6	8	11	14
74°	·2756	·2740	·2723	·2706	·2689	·2672	·2656	·2639	·2622	·2605	3	6	8	11	14
75°	·2588	·2571	·2554	·2538	·2521	·2504	·2487	·2470	·2453	·2436	3	6	8	11	14
76°	·2419	·2402	·2385	·2368	·2351	·2334	·2317	·2300	·2284	·2267	3	6	8	11	14
77°	·2250	·2233	·2215	·2198	·2181	·2164	·2147	·2130	·2113	·2096	3	6	9	11	14
78°	·2079	·2062	·2045	·2028	·2011	·1994	·1977	·1959	·1942	·1925	3	6	9	11	14
79°	·1908	·1891	·1874	·1857	·1840	·1822	·1805	·1788	·1771	·1754	3	6	9	11	14
80°	·1736	·1719	·1702	·1685	·1668	·1650	·1633	·1616	·1599	·1582	3	6	9	11	14
81°	·1564	·1547	·1530	·1513	·1495	·1478	·1461	·1444	·1426	·1409	3	6	9	12	14
82°	·1392	·1374	·1357	·1340	·1323	·1305	·1288	·1271	·1253	·1236	3	6	9	12	14
83°	·1219	·1201	·1184	·1167	·1149	·1132	·1115	·1097	·1080	·1063	3	6	9	12	14
84°	·1045	·1028	·1011	·0993	·0976	·0958	·0941	·0924	·0906	·0889	3	6	9	12	14
85°	·0872	·0854	·0837	·0819	·0802	·0785	·0767	·0750	·0732	·0715	3	6	9	12	14
86°	·0698	·0680	·0663	·0645	·0628	·0610	·0593	·0576	·0558	·0541	3	6	9	12	15
87°	·0523	·0506	·0488	·0471	·0454	·0436	·0419	·0401	·0384	·0366	3	6	9	12	15
88°	·0349	·0332	·0314	·0297	·0279	·0262	·0244	·0227	·0209	·0192	3	6	9	12	15
89°	·0175	·0157	·0140	·0122	·0105	·0087	·0070	·0052	·0035	·0017	3	6	9	12	15

NATURAL TANGENTS.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	3	6	9	12	15
1°	0·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	3	6	9	12	15
2°	0·0349	·0367	·0384	·0402	·0419	·0437	·0454	·0472	·0489	·0507	3	6	9	12	15
3°	0·0524	·0542	·0559	·0577	·0594	·0612	·0629	·0647	·0664	·0682	3	6	9	12	15
4°	0·0699	·0717	·0734	·0752	·0769	·0787	·0805	·0822	·0840	·0857	3	6	9	12	15
5°	0·0875	·0892	·0910	·0928	·0945	·0963	·0981	·0998	·1016	·1033	3	6	9	12	15
6°	0·1051	·1069	·1086	·1104	·1122	·1139	·1157	·1175	·1192	·1210	3	6	9	12	15
7°	0·1228	·1246	·1263	·1281	·1299	·1317	·1334	·1352	·1370	·1388	3	6	9	12	15
8°	0·1405	·1423	·1441	·1459	·1477	·1495	·1512	·1530	·1548	·1566	3	6	9	12	15
9°	0·1584	·1602	·1620	·1638	·1655	·1673	·1691	·1709	·1727	·1745	3	6	9	12	15
10°	0·1763	·1781	·1799	·1817	·1835	·1853	·1871	·1890	·1908	·1926	3	6	9	12	15
11°	0·1944	·1962	·1980	·1998	·2016	·2035	·2053	·2071	·2089	·2107	3	6	9	12	15
12°	0·2126	·2144	·2162	·2180	·2199	·2217	·2235	·2254	·2272	·2290	3	6	9	12	15
13°	0·2309	·2327	·2345	·2364	·2382	·2401	·2419	·2438	·2456	·2475	3	6	9	12	15
14°	0·2493	·2512	·2530	·2549	·2568	·2586	·2605	·2623	·2642	·2661	3	6	9	12	16
15°	0·2679	·2698	·2717	·2736	·2754	·2773	·2792	·2811	·2830	·2849	3	6	9	13	16
16°	0·2867	·2886	·2905	·2924	·2943	·2962	·2981	·3000	·3019	·3038	3	6	9	13	16
17°	0·3057	·3076	·3096	·3115	·3134	·3153	·3172	·3191	·3211	·3230	3	6	10	13	16
18°	0·3249	·3269	·3288	·3307	·3327	·3346	·3365	·3385	·3404	·3424	3	6	10	13	16
19°	0·3443	·3463	·3482	·3502	·3522	·3541	·3561	·3581	·3600	·3620	3	7	10	13	16
20°	0·3640	·3659	·3679	·3699	·3719	·3739	·3759	·3779	·3799	·3819	3	7	10	13	17
21°	0·3839	·3859	·3879	·3899	·3919	·3939	·3959	·3979	·4000	·4020	3	7	10	13	17
22°	0·4040	·4061	·4081	·4101	·4122	·4142	·4163	·4183	·4204	·4224	3	7	10	14	17
23°	0·4245	·4265	·4286	·4307	·4327	·4348	·4369	·4390	·4411	·4431	3	7	10	14	17
24°	0·4452	·4473	·4494	·4515	·4536	·4557	·4578	·4599	·4621	·4642	4	7	11	14	18
25°	0·4663	·4684	·4706	·4727	·4748	·4770	·4791	·4813	·4834	·4856	4	7	11	14	18
26°	0·4877	·4899	·4921	·4942	·4964	·4986	·5008	·5029	·5051	·5073	4	7	11	15	18
27°	0·5095	·5117	·5139	·5161	·5184	·5206	·5228	·5250	·5272	·5295	4	7	11	15	18
28°	0·5317	·5340	·5362	·5384	·5407	·5430	·5452	·5475	·5498	·5520	4	8	11	15	19
29°	0·5543	·5566	·5589	·5612	·5635	·5658	·5681	·5704	·5727	·5750	4	8	12	15	19
30°	0·5774	·5797	·5820	·5844	·5867	·5890	·5914	·5938	·5961	·5985	4	8	12	16	20
31°	0·6009	·6032	·6056	·6080	·6104	·6128	·6152	·6176	·6200	·6224	4	8	12	16	20
32°	0·6249	·6273	·6297	·6322	·6346	·6371	·6395	·6420	·6445	·6469	4	8	12	16	20
33°	0·6494	·6519	·6544	·6569	·6594	·6619	·6644	·6669	·6694	·6720	4	8	13	17	21
34°	0·6745	·6771	·6796	·6822	·6847	·6873	·6899	·6924	·6950	·6976	4	9	13	17	21
35°	0·7002	·7028	·7054	·7080	·7107	·7133	·7159	·7186	·7212	·7239	4	9	13	18	22
36°	0·7265	·7292	·7319	·7346	·7373	·7400	·7427	·7454	·7481	·7508	5	9	14	18	23
37°	0·7536	·7563	·7590	·7618	·7646	·7673	·7701	·7729	·7757	·7785	5	9	14	18	23
38°	0·7813	·7841	·7869	·7898	·7926	·7954	·7983	·8012	·8040	·8069	5	9	14	19	24
39°	0·8098	·8127	·8156	·8185	·8214	·8243	·8273	·8302	·8332	·8361	5	10	15	20	24
40°	0·8391	·8421	·8451	·8481	·8511	·8541	·8571	·8601	·8632	·8662	5	10	15	20	25
41°	0·8693	·8724	·8754	·8785	·8816	·8847	·8878	·8910	·8941	·8972	5	10	16	21	26
42°	0·9004	·9036	·9067	·9099	·9131	·9163	·9195	·9228	·9260	·9293	5	11	16	21	27
43°	0·9325	·9358	·9391	·9424	·9457	·9490	·9523	·9556	·9590	·9623	6	11	17	22	28
44°	0·9657	·9691	·9725	·9759	·9793	·9827	·9861	·9896	·9930	·9965	6	11	17	23	29

NATURAL TANGENTS.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	1·0000	1·0035	1·0070	1·0105	1·0141	1·0176	1·0212	1·0247	1·0283	1·0319	6	12	18	24	30
46°	1·0355	1·0392	1·0428	1·0464	1·0501	1·0538	1·0575	1·0612	1·0649	1·0686	6	12	18	25	31
47°	1·0724	1·0761	1·0799	1·0837	1·0875	1·0913	1·0951	1·0990	1·1028	1·1067	6	13	19	25	32
48°	1·1106	1·1145	1·1184	1·1224	1·1263	1·1303	1·1343	1·1383	1·1423	1·1463	7	13	20	26	33
49°	1·1504	1·1544	1·1585	1·1626	1·1667	1·1708	1·1750	1·1792	1·1833	1·1875	7	14	21	28	34
50°	1·1918	1·1960	1·2002	1·2045	1·2088	1·2131	1·2174	1·2218	1·2261	1·2305	7	14	22	29	36
51°	1·2349	1·2393	1·2437	1·2482	1·2527	1·2572	1·2617	1·2662	1·2708	1·2753	8	15	23	30	38
52°	1·2799	1·2846	1·2892	1·2938	1·2985	1·3032	1·3079	1·3127	1·3175	1·3222	8	16	24	31	39
53°	1·3270	1·3319	1·3367	1·3416	1·3465	1·3514	1·3564	1·3613	1·3663	1·3713	8	16	25	33	41
54°	1·3764	1·3814	1·3865	1·3916	1·3968	1·4019	1·4071	1·4124	1·4176	1·4229	9	17	26	34	43
55°	1·4281	1·4335	1·4388	1·4442	1·4496	1·4550	1·4605	1·4659	1·4715	1·4770	9	18	27	36	45
56°	1·4826	1·4882	1·4938	1·4994	1·5051	1·5108	1·5166	1·5224	1·5282	1·5340	10	19	29	38	48
57°	1·5399	1·5458	1·5517	1·5577	1·5637	1·5697	1·5757	1·5818	1·5880	1·5941	10	20	30	40	50
58°	1·6003	1·6066	1·6128	1·6191	1·6255	1·6319	1·6383	1·6447	1·6512	1·6577	11	21	32	43	53
59°	1·6643	1·6709	1·6775	1·6842	1·6909	1·6977	1·7045	1·7113	1·7182	1·7251	11	23	34	45	56
60°	1·7321	1·7391	1·7461	1·7532	1·7603	1·7675	1·7747	1·7820	1·7893	1·7966	12	24	36	48	60
61°	1·8040	1·8115	1·8190	1·8265	1·8341	1·8418	1·8495	1·8572	1·8650	1·8728	13	26	38	51	64
62°	1·8807	1·8887	1·8967	1·9047	1·9128	1·9210	1·9292	1·9375	1·9458	1·9542	14	27	41	55	68
63°	1·9626	1·9711	1·9797	1·9883	1·9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64°	2·0503	2·0594	2·0686	2·0778	2·0872	2·0965	2·1060	2·1155	2·1251	2·1348	16	31	47	63	78
65°	2·1445	2·1543	2·1642	2·1742	2·1842	2·1943	2·2045	2·2148	2·2251	2·2355	17	34	51	68	85
66°	2·2460	2·2566	2·2673	2·2781	2·2889	2·2998	2·3109	2·3220	2·3332	2·3445	18	37	55	73	92
67°	2·3559	2·3673	2·3789	2·3906	2·4023	2·4142	2·4262	2·4383	2·4504	2·4627	20	40	60	79	99
68°	2·4751	2·4876	2·5002	2·5129	2·5257	2·5386	2·5517	2·5649	2·5782	2·5916	22	43	65	87	108
69°	2·6051	2·6187	2·6325	2·6464	2·6605	2·6746	2·6889	2·7034	2·7179	2·7326	24	47	71	95	119
70°	2·7475	2·7625	2·7776	2·7929	2·8083	2·8239	2·8397	2·8556	2·8716	2·8878	26	52	78	104	130
71°	2·9042	2·9208	2·9375	2·9544	2·9714	2·9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	144
72°	3·0777	3·0961	3·1146	3·1334	3·1524	3·1716	3·1910	3·2106	3·2305	3·2506	32	64	96	129	161
73°	3·2709	3·2914	3·3122	3·3332	3·3544	3·3759	3·3977	3·4197	3·4420	3·4646	36	72	108	144	180
74°	3·4874	3·5105	3·5339	3·5576	3·5816	3·6059	3·6305	3·6554	3·6806	3·7062	41	81	122	163	204
75°	3·7321	3·7583	3·7848	3·8118	3·8391	3·8667	3·8947	3·9232	3·9520	3·9812					
76°	4·0108	4·0408	4·0713	4·1022	4·1335	4·1653	4·1976	4·2303	4·2635	4·2972					
77°	4·3315	4·3662	4·4015	4·4374	4·4737	4·5107	4·5483	4·5864	4·6252	4·6646					
78°	4·7046	4·7453	4·7867	4·8288	4·8716	4·9152	4·9594	5·0045	5·0504	5·0970					
79°	5·1446	5·1929	5·2422	5·2924	5·3435	5·3955	5·4486	5·5026	5·5578	5·6140					
80°	5·6713	5·7297	5·7894	5·8502	5·9124	5·9758	6·0405	6·1066	6·1742	6·2432	Mean differences not sufficiently accurate.				
81°	6·3138	6·3859	6·4596	6·5350	6·6122	6·6912	6·7720	6·8548	6·9395	7·0264					
82°	7·1154	7·2066	7·3002	7·3962	7·4947	7·5958	7·6996	7·8062	7·9158	8·0285					
83°	8·1443	8·2636	8·3863	8·5126	8·6427	8·7769	8·9152	9·0579	9·2052	9·3572					
84°	9·5144	9·6768	9·8448	10·019	10·199	10·385	10·579	10·780	10·988	11·205					
85°	11·430	11·664	11·909	12·163	12·429	12·706	12·996	13·300	13·617	13·951					
86°	14·301	14·669	15·056	15·464	15·895	16·350	16·832	17·343	17·886	18·464					
87°	19·081	19·740	20·446	21·205	22·022	22·904	23·859	24·898	26·031	27·271					
88°	28·636	30·145	31·821	33·694	35·801	38·188	40·917	44·066	47·740	52·081					
89°	57·290	63·657	71·615	81·847	95·489	114·59	143·24	190·98	286·48	572·96					

LOGARITHMS OF SINES.

Angle	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	$-\infty$	$\bar{3}$·242	$\bar{3}$·543	$\bar{3}$·719	$\bar{3}$·844	·941	$\bar{2}$·020	$\bar{2}$·087	$\bar{2}$·145	$\bar{2}$·196	Differences not sufficiently accurate.				
1°	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2°	$\bar{2}$·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3°	$\bar{2}$·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4°	$\bar{2}$·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315					
5°	$\bar{2}$·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{1}$·0046	$\bar{1}$·0120	13	26	39	52	65
6°	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7°	$\bar{1}$·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8°	$\bar{1}$·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9°	$\bar{1}$·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10°	$\bar{1}$·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11°	$\bar{1}$·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12°	$\bar{1}$·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13°	$\bar{1}$·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14°	$\bar{1}$·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15°	$\bar{1}$·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16°	$\bar{1}$·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17°	$\bar{1}$·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18°	$\bar{1}$·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19°	$\bar{1}$·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20°	$\bar{1}$·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21°	$\bar{1}$·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22°	$\bar{1}$·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23°	$\bar{1}$·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24°	$\bar{1}$·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25°	$\bar{1}$·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26°	$\bar{1}$·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27°	$\bar{1}$·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28°	$\bar{1}$·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29°	$\bar{1}$·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30°	$\bar{1}$·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31°	$\bar{1}$·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32°	$\bar{1}$·7242	7254	7266	7273	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33°	$\bar{1}$·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34°	$\bar{1}$·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35°	$\bar{1}$·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36°	$\bar{1}$·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37°	$\bar{1}$·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38°	$\bar{1}$·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39°	$\bar{1}$·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40°	$\bar{1}$·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41°	$\bar{1}$·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42°	$\bar{1}$·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43°	$\bar{1}$·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44°	$\bar{1}$·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

LOGARITHMS OF SINES.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46°	$\bar{1}$·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47°	$\bar{1}$·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48°	$\bar{1}$·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49°	$\bar{1}$·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50°	$\bar{1}$·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51°	$\bar{1}$·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52°	$\bar{1}$·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53°	$\bar{1}$·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54°	$\bar{1}$·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55°	$\bar{1}$·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56°	$\bar{1}$·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57°	$\bar{1}$·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58°	$\bar{1}$·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59°	$\bar{1}$·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60°	$\bar{1}$·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61°	$\bar{1}$·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62°	$\bar{1}$·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63°	$\bar{1}$·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64°	$\bar{1}$·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65°	$\bar{1}$·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66°	$\bar{1}$·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67°	$\bar{1}$·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68°	$\bar{1}$·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69°	$\bar{1}$·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70°	$\bar{1}$·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71°	$\bar{1}$·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72°	$\bar{1}$·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73°	$\bar{1}$·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74°	$\bar{1}$·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75°	$\bar{1}$·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76°	$\bar{1}$·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77°	$\bar{1}$·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78°	$\bar{1}$·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79°	$\bar{1}$·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80°	$\bar{1}$·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81°	$\bar{1}$·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82°	$\bar{1}$·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83°	$\bar{1}$·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84°	$\bar{1}$·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85°	$\bar{1}$·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86°	$\bar{1}$·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87°	$\bar{1}$·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88°	$\bar{1}$·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89°	$\bar{1}$·9999	9999	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0	0	0	0	0

LOGARITHMS OF COSINES.

Subtract Differences.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	9999	0	0	0	0	0
1°	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2°	$\bar{1}$·9997	9997	9997	9997	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3°	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4°	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5°	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6°	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7°	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8°	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9°	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10°	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11°	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12°	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13°	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14°	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15°	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16°	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17°	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18°	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19°	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20°	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21°	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22°	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23°	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24°	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25°	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26°	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27°	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28°	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29°	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30°	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31°	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32°	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33°	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34°	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35°	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36°	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37°	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38°	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39°	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40°	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41°	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42°	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43°	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44°	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

LOGARITHMS OF COSINES.

Subtract Differences

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46°	$\bar{1}$·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47°	$\bar{1}$·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48°	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49°	$\bar{1}$·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50°	$\bar{1}$·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51°	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52°	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53°	$\bar{1}$·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54°	$\bar{1}$·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55°	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56°	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57°	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58°	$\bar{1}$·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59°	$\bar{1}$·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60°	$\bar{1}$·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61°	$\bar{1}$·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62°	$\bar{1}$·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63°	$\bar{1}$·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64°	$\bar{1}$·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65°	$\bar{1}$·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66°	$\bar{1}$·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67°	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68°	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69°	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70°	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71°	$\bar{1}$·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72°	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73°	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74°	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75°	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76°	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77°	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78°	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79°	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80°	$\bar{1}$·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81°	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82°	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83°	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84°	$\bar{1}$·0192	0120	0046	$\bar{2}$·9970	$\bar{2}$·9894	$\bar{2}$·9816	$\bar{2}$·9736	$\bar{2}$·9655	$\bar{2}$·9573	$\bar{2}$·9489	13	26	39	52	66
85°	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86°	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87°	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88°	$\bar{2}$·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89°	$\bar{2}$·2419	$\bar{2}$·196	$\bar{2}$·145	$\bar{2}$·087	$\bar{2}$·020	$\bar{3}$·941	$\bar{3}$·844	$\bar{3}$·719	$\bar{3}$·543	$\bar{3}$·242					

LOGARITHMS OF TANGENTS.

Angle	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	$-\infty$	$\bar{3}$·242	$\bar{3}$·543	$\bar{3}$·719	$\bar{3}$·844	$\bar{3}$·941	$\bar{2}$·020	$\bar{2}$·087	$\bar{2}$·145	$\bar{2}$·196					
1°	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2°	$\bar{2}$·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3°	$\bar{2}$·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4°	$\bar{2}$·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5°	$\bar{2}$·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{1}$·0068	$\bar{1}$·0143	13	26	40	53	66
6°	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7°	$\bar{1}$·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8°	$\bar{1}$·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9°	$\bar{1}$·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10°	$\bar{1}$·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11°	$\bar{1}$·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12°	$\bar{1}$·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13°	$\bar{1}$·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14°	$\bar{1}$·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15°	$\bar{1}$·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16°	$\bar{1}$·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17°	$\bar{1}$·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18°	$\bar{1}$·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19°	$\bar{1}$·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20°	$\bar{1}$·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21°	$\bar{1}$·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22°	$\bar{1}$·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23°	$\bar{1}$·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24°	$\bar{1}$·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25°	$\bar{1}$·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26°	$\bar{1}$·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27°	$\bar{1}$·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28°	$\bar{1}$·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29°	$\bar{1}$·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30°	$\bar{1}$·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31°	$\bar{1}$·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32°	$\bar{1}$·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33°	$\bar{1}$·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34°	$\bar{1}$·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35°	$\bar{1}$·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36°	$\bar{1}$·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37°	$\bar{1}$·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38°	$\bar{1}$·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39°	$\bar{1}$·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40°	$\bar{1}$·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41°	$\bar{1}$·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42°	$\bar{1}$·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43°	$\bar{1}$·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44°	$\bar{1}$·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

LOGARITHMS OF TANGENTS.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	0·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46°	0·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47°	0·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48°	0·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49°	0·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50°	0·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51°	0·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52°	0·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53°	0·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54°	0·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55°	0·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56°	0·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57°	0·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58°	0·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59°	0·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60°	0·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61°	0·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62°	0·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63°	0·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64°	0·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	7	10	13	16
65°	0·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66°	0·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67°	0·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68°	0·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69°	0·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70°	0·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71°	0·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72°	0·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73°	0·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74°	0·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75°	0·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76°	0·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77°	0·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78°	0·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79°	0·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80°	0·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81°	0·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82°	0·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83°	0·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84°	0·9784	9857	9932	1·0008	1·0085	1·0164	1·0244	1·0326	1·0409	1·0494	13	26	40	53	66
85°	1·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	65	81
86°	1·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87°	1·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88°	1·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89°	1·7581	1·804	1·855	1·913	1·980	2·059	2·156	2·281	2·457	2·758					

ANSWERS

EXERCISE 1

p. 16.

1. 4, 125, $\sqrt{8}$. **2.** $\frac{1}{4}$, $\frac{1}{25}$, 100, 16. **3.** 4, $3\frac{3}{8}$, 4.
4. $\frac{1}{8}$, 8, $\frac{1}{32}$. **5.** 5·196, 15·588, 2·828.
6. 31 62, 0·01, 0·1, 0·3162. **7.** 1·19, 2·38.
8. 0·125, 0·0442. **9.** $a^{\frac{5}{2}}$ or $\sqrt[2]{a^5}$. **10.** $\frac{1}{\sqrt[6]{a}}$.
11. 4·64. **12.** $9a^2b$, $2a^2b$.

EXERCISE 2

p. 25.

1. 0·4831. **2.** 0·0028. **3.** 0·07936. **4.** 0·8265.
5. 3109. **6.** 0·3423. **7.** 0·9330. **8.** 0·7071.
9. 2·174. **10.** 0·2026. **11.** 2·048. **12.** 0·9906.
13. 54·67. **14.** 0·2394. **15.** 0·8246. **16.** 5·081.
17. 4·357. **18.** 5·119. **19.** 0·9913. **20.** 10·43.
21. 80·09. **22** (*a*) 318·6; (*b*) 2·463; (*c*) 1·406.
23. 0·00848. **24.** 4. **25.** 48·17. **26.** 42·94.
27. 6·165. **28.** 0·0294. **29.** 0·3059. **30.** 1·213.
31. 249·7. **32.** 5·888.
33. (i) $10^{\frac{1}{2}}$, $10^{-0·0809}$, $10^{0·4202}$; (ii) $x = 2·431$; (iii) 10·27.
34. (i) $10^{-0·42}$, $10^{0·14}$; (ii) $x = 1·177$; (iii) 0·7886.
35. $T_1 = 336·8$, $T_2 = 176·8$, $\frac{T_1}{T_2}$ is squared.

EXERCISE 3

p. 33.

1. (*a*) 3; (*b*) 3; (*c*) 6; (*d*) 1·5; (*e*) 2·5; (*f*) 0·5.
2. (*a*) 2·431; (*b*) 2·460.
3. (*a*) 1·5261; (*b*) 2·0149; (*c*) 2·2618; (*d*) 3·688; (*e*) 4·024; (*f*) $\bar{3}$·2195 (or − 2·7805).
5. (*a*) 4·015; (*b*) 6·424; (*c*) 9·07; (*d*) 0·4299.
6. 0·9163. **7.** 1·6. **8.** 0·09509. **9.** $\bar{1}$·4892.

10. 5·0005. **11.** 650. **12.** 2·4868.
13. (*a*) 5; (*b*) 36; (*c*) 12; (*d*) 16; (*e*) x; (*f*) e.
14. 100 ft. **16.** 0·8268. **17.** 76. **18.** 22745.
19. 686·3. **20.** 39·76. **21.** (*a*) 1·946; (*b*) 86·22.

EXERCISE 4

p. 43.

1. − 3, 8. **2.** $\frac{2}{3}$, $\frac{3}{2}$. **3.** 2, − $\frac{5}{3}$. **4.** $\frac{1}{4}$, − $\frac{1}{6}$.
5. $\frac{1}{3}$, 0. **6.** $\frac{1}{2}$, − $\frac{1}{5}$. **7.** 1·531, − 6·531.
8. 2·591, − 0·257. **9.** − 0·1209, − 0·7702.
10. 3·18, 0·43. **11.** 17·77, − 3·77. **12.** 9·75, − 1·75.
13. 3, 2, − 1. **14.** $0·5 \pm 0·866i$. **15.** $0·2 \pm 1·077i$.
16. $- 1·166 \pm 1·143i$. **17.** $6 \pm 2i$.
18. $0·333 \pm 1·247i$. **19.** $(x - 1·8)(x + 2·9)$.
20. $3(x - 2·6)(x + 1·9)$. **21.** 12 ins., 9 ins.
22. $10\frac{1}{2}$ ins., $8\frac{1}{2}$ ins. **23.** 18 or 2.
24. 6·545 or 0·955 secs. **25.** 24.
26. 3·985 ins. **27.** 1·35 p.m. **28.** 89·82. **29.** 7·5.
30. 5·14. **31.** 0·04555.

EXERCISE 5

p. 48.

1. $x = 6, y = 5$; $x = -5, y = -6$.
2. $x = 5, y = 4$; $x = -5, y = -11$.
3. $x = -5, y = 4$; $x = 5, y = -4$.
4. $x = 3, y = -2$; $x = 2, y = -1$.
5. $x = -\frac{12}{5}, y = \frac{17}{5}$; $x = 3, y = -2$.
6. $x = \frac{4}{3}, y = \frac{34}{9}$; $x = 1, y = 4$.
7. $x = 6, y = 2$; $x = -4, y = -3$.
8. $x = 4, y = 1$; $x = -1, y = -4$.
9. $x = 8, y = 5$; $x = -8, y = -5$; $x = 5, y = 8$;
$x = -5, y = -8$.
10. $x = -\frac{19}{8}, y = -\frac{33}{2}$; $x = 3, y = 2$.
11. $x = 3, y = -\frac{1}{3}$; $x = 5, y = -1$.
12. $x = \frac{1}{2}, y = \frac{3}{4}$; $x = 1\frac{1}{2}, y = -\frac{1}{4}$.
13. (i) − 2·06, 0·728; (ii) − 3, 2; (iii) $5\frac{1}{2}$, − $\frac{5}{6}$.

EXERCISE 6

p. 56.

1. (*a*) 0·1, − 4·1; (*b*) 2·85, 0·15.
2. (*a*) − 1·454, 1·204; (*b*) 0, 0; 118, − 47200.
3. 4·828, 0·828. **4.** 3, − $\frac{1}{2}$. **5.** $\frac{1}{2}$, 2. **6.** $1\frac{1}{3}$.
7. 2, − 1, 3, 19. **8.** 0·24, 2; − 1·4, − 1·4.

EXERCISE 7

p. 64.

1. 72·36 sq. ins. **2.** $40\frac{1}{2}°$. **3.** 39° 1′.
4. 64·8 sq. ins. **5.** 361·3 sq. chs. **6.** 42·33 sq. mls.
7. 180·4 sq. ft. **8.** 3038 sq. mls. **9.** 31·44 lb.
10. 1·11 oz. **11.** 247·7 sq. ins. **12.** 173·8 sq. ins.
13. 60 ins, 259·8 sq. ins. **14.** 73·47 sq. ins., 30·9 ins.
15. 82·83 sq. ins., 33·13 ins. **16.** 21·4 sq. ins., 17·63 ins.
17. 2·6 sq. ins.
18. 11·196 sq. ft., 14·39 ft. (top included), 8·928 (without top) $H = 1·254$.

EXERCISE 8

p. 79.

1. 260 sq. ins. **2.** 60·67 sq. ins.; 6·067 ins.
3. 4·275. **4.** 7·854 cms.
5. Pressure = 11·3 lb. per sq. in.; Work = 11·3 ft.-lb.
6. Speed = 32 ft. per sec.; Distance = 1920 ft.
7. 32,000,000 gals. **8.** 2187 cub. yds.
9. 900 lb. **10.** 1050 tons.
11. Vol. = 7295 cub. ft.; Wt. = 162·1 tons.
12. Area = 9·22 sq. ft.; Wt. = 12 lb. 7 oz.
13. 40·12 sq. ins.

EXERCISE 9

p. 95.

1. 3·56 ins., 0·44 ins. **2.** 118°, 3·6 ins., 7·4 ins.
3. 19·8 ins., 23·9 ins. **4.** 3·39 sq. ins., 1·41 sq. ins.
5. 118·5 gms. **6.** 201°. **7.** $\omega = 31·4$, $v = 5·23$.
8. $4\frac{1}{2}$, 57·3. **9.** $\frac{L^2}{8d}$.
10. 70·686 sq. ft., 3 and $4\frac{1}{2}$ ft., 712·5 lb. **11.** $w = \frac{12(q-p)}{c} = 3$.
12. 0·733 rads. = 42°; 2·409 rads. = 138°.
13. 223·045, ± 0·3984. **14.** 5·89 ft., 6·51 ft.

EXERCISE 10

p. 113.

1. 98 sq. ft. **2.** 0·145 in.
3. 363·75 cub. ins., 206·25 sq. ins.
4. 8·29 ins. **5.** 30 lb. **6.** 4·687 ins. **7.** 9·718 ins.
8. 440 gals. **9.** 3·61 cms. **10.** 35 cub. ins.
11. 15 sq. ins.; 65 sq. ins.

EXERCISE 11

p. 121.

1. -15. **2.** 8. **3.** 3·48 **4.** $1\frac{7}{12}$.
5. $5n - 3$. **6.** $8 - 2p$. **7.** 3. **8.** 31.
9. 0·1. **10.** 12th. **11.** 11th. **12.** 20, $16\frac{4}{11}$, $12\frac{8}{11}$
13. -6, 0, 6, etc **14.** 0·19, 0·22, 0·25, 0·28.
15. $\frac{3x + y}{4}$, $\frac{x + y}{2}$, $\frac{x + 3y}{4}$. **16.** 87. **17.** -5.
18. $-74·9$. **19.** 192. **20.** 210. **21.** 11.
22. 624 ft., 6400 ft., 16 secs. **23.** £13 15*s*., 16 yrs.
24. £2700, 11 yrs. **25.** £228 18*s*. $1\frac{1}{2}$*d*., £1 13*s*. $10\frac{1}{2}$.
26. (*a*) first by £32; (*b*) second by £36.
27. 19 rows (10 over). **28.** $19\frac{1}{2}$ m.p.h.
29. 973. **30.** 120, 1860.

EXERCISE 12

p. 183.

1. 2·8561. **2.** $-\frac{25}{256}$. **3.** 2·358. **4.** 0·001093
5. $-0·002147$. **6.** 20. **7.** 8·538. **8.** 0·6277. **9.** 80, 64
10. 60, 48, $38\frac{2}{5}$. **11.** 16·32. **12.** 19·93. **13.** $9\frac{13}{32}$.
14. 216·8. **15.** $3\frac{23}{41}$. **16.** $-4·918$. **17.** 45.
18. $10\frac{2}{3}$. **19.** $5\frac{1}{3}$. **20.** $\frac{5}{9}$, $\frac{4}{11}$, $3\frac{5}{8}$. **21.** 90 ft.
22. 70 ft. **23.** £1347 12*s*. **24.** £75,000. **25.** 626 ft.

EXERCISE 13

p. 156.

1. (*a*) 0·9781, $-0·2079$, $-4·7046$;
(*b*) $-0·8873$, $-0·4612$, 1·9237;
(*c*) $-0·7030$, 0·7112, $-0·9884$;
(*d*) 0, 1, 0; (*e*) 1, 0, ∞;
(*f*) $-0·0610$, 0·9981, $-0·0612$;
(*g*) $-0·7771$, 0·6293, $-1·2349$;
(*h*) 0·866, 0·5000, 1·7321;
(*i*) 0·9703, $-0·2419$, $-4·0108$;
(*j*) $-0·3420$, $-0·9397$, 0·3640;
(*k*) 0·5068, $-0·8621$, $-0·5879$;
(*l*) $-0·3619$, $-0·9322$, 0·3882;
(*m*) $-0·2901$, 0·9570, $-0·3032$.
2. (*a*) 2·281; (*b*) $-1·744$; (*c*) $-1·235$; (*d*) $-1·058$;
(*e*) 2; (*f*) $-3·271$.
3. $-0·9165$, 0·4365. **4.** 94° 8′, 265° 52′.

5. 28° 41′, 151° 19′. **6.** $\sec\theta = \mp\frac{13}{12}$, $\cot\theta = \pm\frac{12}{5}$.
7. 0·9798, 0·2041. **8.** $-$ 1·732. **9.** $\frac{5}{13}$.
10. $\frac{1}{\sqrt{1-m^2}}, \frac{\sqrt{1-m^2}}{m}$. **11.** $-$ 4·428.
12. 0. **13.** 0·966. **14.** 0·2588. **15.** 0·9511.
25. 26°, 110°. **26.** 28°, 120°. **27.** $-$ 611·6, 1199·0, 547·1.
28. 55°.

EXERCISE 14

p. 170.

1. c = 7·881, b = 5·588, C = 56°.
2. a = 4·879, B = 57° 28′, C = 62° 32′.
3. b = 24·68 yds., C = 47° 28′., A = 72° 32′.
4. B = 48° 24′, A = 71° 36′, a = 241·1.
5. c = 40·92, A = 65° 18′, B = 36° 42′.
6. A = 28° 54′, B = 32°, C = 119° 6′.
7. B = 64°, c = 596·2 yds., b = 586·6 yds.
8. B = 70° or 110°, C = 60° or 20°.
10. 64° 31′. **11.** 13·72 ins. **12.** 19·05 sq. ins.
14. 160 m.p.h. **15.** 16·35, 13·62. **16.** 60°.
17. 8·324 tons, 23°. **18.** 989·3 ft. per sec., 140° 11½′.
19. 11·62 sq. ins., 104°. **20.** (b) 41°. **21.** 4⅛ hrs.
22. 4 ins., 1·6 ins.; 81° 13′, 36° 52′, 61° 55′.
23. 4·5 ins., 6 ins.; 11 sq. ins.

EXERCISE 15

p. 188.

1. 0·94, 0·34. **2.** 0·93. **3.** 0·8545.
4. 0·8945, $-$ 2. **5.** 3·0777, 0·5407. **6.** 0·3919.
7. 0·477. **8.** 0·96, 0·28, 3·428.
9. 0·9917, $-$ 0·1288. **10.** 0·6151, 0·3253.
11. 4·5 sin $(6t + 0{\cdot}927)$. **12.** 39·04 sin $(2\pi nt + 0{\cdot}6946)$.
13. 5·83 sin $(4t - 0{\cdot}54)$. **32.** 0°, 180°, 80° 25′, 279° 35′.
33. 26° 34′, 45°, 206° 34′, 225°. **34.** 0°, 90°, 180°.
35. 60°, 120°, 240°, 300°. **36.** 43° 52′, 136° 8′.
37. 60°, 300°. **38.** 26° 35′, 153° 25′, 206° 35′, 333° 25′.
39. 45°, 135°, 225°, 315°. **40.** 60°, 120°, 240°, 300°.
41. 10° 33′, 259° 27′. **42.** 30°, 150°, 210°, 330°.
43. 78° 28′, 281° 32′. **44.** 45°, 135°, 225°, 315°.
45. 0°, 120°, 180°, 240°. **46.** 0°, 60°, 180°, 300°.
47. 0°, 120°, 240°. **48.** 180°.

EXERCISE 16

p. 203.

1. (a) $x = -4\cdot5, 1\cdot67$; (b) $x = -3\cdot9, 1\cdot06$; (c) Min. 57.
2. $x = -4\cdot5, 1\cdot67$. **3.** Max. value is $8\cdot7$ when $x = 2\cdot4$.
4. $r = 5\cdot4$, $N = 20\cdot13$. **5.** Min. value is $48\cdot5$, when $M = 2\cdot32$.
6. $R = 0\cdot32$.
7. $x = 0, 3, -4$; max. value $41\cdot5$, min. value $-25\cdot2$.
8. $x = 2\cdot8$. **9.** $x = 3\cdot532$. **10.** $-1\cdot4, 0\cdot65, 2\cdot75$.
11. $x = 3\frac{1}{2}$ gives max. value of 180.
12. (a) 96 when speed-ratio is 5; (b) $0\cdot715$; (c) 2.
13. $-0\cdot57, 0\cdot75, 2\cdot3$. **14.** $-2\cdot7, 0\cdot55, 2\cdot15$.
15. $-1\cdot4, 0\cdot8, 3\cdot6$.
16. (1) $-1\cdot7, 2\cdot5, 5\cdot2$; (2) $-1\cdot4, 1\cdot9, 5\cdot5$.
17. $54\cdot5$. **18.** $3\cdot65$.

EXERCISE 17

p. 221.

1. $a = 2\cdot3, b = -10$.
2. $L = 606\cdot5 - 0\cdot695\theta$; $536\cdot5$.
3. Missing values of (a) x . . . $0\cdot6$; (b) y . . . $-8, -3, 4, 22$ (The law is $y = 5x - 3$).
4. $E = 0\cdot45W$. **5.** $m = 5\cdot3, C = 10\cdot5$.
6. Missing values of (a) x . . . $0, 1\cdot2$; (b) y . . . $8, 1\cdot6$ ($a = 8, b = 2$).
7. $a = 0\cdot42, b = 7\cdot5$. **8.** $a = 40, b = 0\cdot03$.
9. $P = 1\cdot2S^2 + 1$. **10.** $a = 0\cdot3, b = 0\cdot01$.
11. $P = 1\cdot08V^3 + 420$. **12.** $a = 0\cdot003, b = 0\cdot15$.
13. $a = 210, n = 0\cdot3$. **14.** $a = 5\cdot7, n = 0\cdot36$.
15. $a = 16, \mu = 0\cdot3$. **16.** $a = 0\cdot88, b = 0\cdot4$.
17. $A = 30, k = -0\cdot000039$. **18.** 270.
19. $n = 3\cdot28, a = 0\cdot56$. **20.** $a = 0\cdot54, b = 0\cdot24$.
21. $P = 31\cdot6v^{1\cdot78}$. **22.** $a = 100, b = 1\cdot25, s = \frac{1}{100m^{0\cdot25}}$.

EXERCISE 18

p. 232.

1. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$.
2. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$.
3. $x^6 - 12x^4 + 60x^2 - 160 + \frac{240}{x^2} - \frac{192}{x^4} + \frac{64}{x^6}$.
4. $1 - 7xy + 21x^2y^2 - 35x^3y^3 + 35x^4y^4 - 21x^5y^5 + 7x^6y^6 - x^7y^7$.
5. $a^8 - 8a^6c + 24a^4c^2 - 32a^2c^3 + 16c^4$.
6. $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

7. $1 - \frac{10}{x} + \frac{45}{x^2} - \frac{120}{x^3} + \frac{210}{x^4} - \frac{252}{x^5} + \frac{210}{x^6} - \frac{120}{x^7} + \frac{45}{x^8} - \frac{10}{x^9} + \frac{1}{x^{10}}$

8. $\frac{1}{256} + \frac{a}{16} + \frac{7a^2}{16} + \frac{7a^3}{4} + \frac{35a^4}{8} + 7a^5 + 7a^6 + 4a^7 + a^8$.

9. $\frac{1}{x^2} - \frac{2\Delta x}{x^3} + \frac{3(\Delta x)^2}{x^4} \ldots$ 10. $\frac{1}{m^4} - \frac{4n}{m^5} + \frac{10n^2}{m^6} \ldots$

11. 1·012. 12. 1·003. 13. 1·000036.

14. $\frac{945}{16}a^3b^4$. 15. $15120x^4y^3$. 16. $\frac{1120}{81}a^4b^4$.

17. $-112640x^9$. 18. $\frac{5940}{x^6}$. 19. $\frac{448}{9y^5}$. 20. $\frac{55x^4b^6}{9}$.

21. $x^{12} - 144x^{11} + 9504x^{10} - 380160x^9 \ldots$

22. $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$.

EXERCISE 19

p. 262.

1. (*a*) 2; (*b*) $10t - 2$; (*c*) $12t^2 + 2t$; (*d*) $-\frac{3}{x^4}$.

2. (*a*) $20x^3 + 3$; (*b*) $18t^2 - 6t + 1$; (*c*) $\frac{3}{2\sqrt{x}}$; (*d*) r;
(*e*) $\frac{4x}{3} - \frac{5}{6} + \frac{2}{x^2} - \frac{8}{x^3}$; (*f*) $\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$;
(*g*) $2{\cdot}76x^{0{\cdot}2} - \frac{1}{\sqrt{x}} - \frac{7}{x^2}$; (*h*) $10 - 6t$, (i) $0{\cdot}405x^{0{\cdot}35}$.

3. (*a*) $6u + 1$; (*b*) $1{\cdot}2x^{0{\cdot}2} + \frac{0{\cdot}2}{x^{1.2}} + 3x^{0{\cdot}5}$; (*c*) $-\frac{560}{v^{3{\cdot}4}}$;
(*d*) $0{\cdot}1125v^{0{\cdot}5}$; (*e*) $-\frac{0{\cdot}001}{h^2}$.

4. $54t^2 + 6t + 14$. 5. $\frac{12 - 3x}{x^3}$. 6. $24t - 18$.

7. 0·07 sq. in. per sec. 8. 22·62 cub. ins. per sec.
9. 8 ins. 10. $+6, -8$. 11. $-2, +1$.
12. 7, 0·25. 13. (*a*) $\frac{2}{3}$ or -1; (*b*) 0 or $-\frac{1}{3}$.
14. 16·97. 15. (*a*) 3·5; (*b*) 2·5; (*c*) 1·75.
16. 23.
17. (*a*) $100 - 8t$ ft. per sec.; (*b*) 68 ft. per sec.; (*c*) acc^{n} = -8.
18. (*a*) 41 ft. per sec.; (*b*) 6 ft. per sec.^{2}; (*c*) $t = 4$ secs.; (*d*) $t = 8{\cdot}2$ secs.
19. 55 ft. per sec., 12 ft. per sec.^{2}.
20. (*a*) 57 ft. per sec.; (*b*) 8·89 secs.; (*c*) 14 ft. per sec.^{2}; (*d*) $4\frac{1}{3}$ secs.

21. (*a*) 16 ft., 32 ft. per sec.; (*b*) $120 - 32t - 16\Delta t$; (*d*) $120 - 32t$, 56 ft. per sec.; (*e*) $3\frac{3}{4}$ secs., 225 ft.
22. (*a*) $3\frac{1}{8}$ secs.; (*b*) 70·72 ft. per sec.
23. (*a*) 50 radians; (*b*) 2 radians per sec.; (*c*) 12 radians per sec.; (*d*) $-$ 2 radians per sec.^{2}; 6 secs.
24. $11 + 3\delta x$; 11 (these are actually gradients).
25. 0·8, 0, $-$ 0·8 (actual slope = 38° 40′, 0°, 141° 20′).
26. (1) 21·17 secs.; (2) 8·47 ft. per sec.
27. $3x^2 - 4x - 11$; 9.
28. (*a*) $\frac{\delta s}{\delta t} = 24 + 4\delta t$, $\frac{ds}{dt} = 8t$, 24 ft. per sec.; (*b*) $6x^2 - 18x - 60$, 0 (these are actually gradients).
29. 200 ft. per sec.; 72 ft. per sec.; $6\frac{1}{4}$ secs.
30. $-$ 5.

EXERCISE 20

p. 287.

1. Min. at $x = \frac{1}{3}$.
2. Min. at $x = -\frac{3}{8}$.
3. Max. at $x = 2\frac{1}{2}$.
4. Min. at $x = -\frac{5}{4}$.
5. Max. at $x = 12$.
6. Min. at $x = 1\frac{1}{2}$.
7. Max. 25 when $x = -1$; min. $-$ 71 when $x = 3$.
8. Max. 0 when $x = 1$; min. $-$ 32 when $x = 5$.
9. Min. $-$ $18\frac{19}{27}$ when $x = -\frac{4}{3}$; max. 63 when $x = 3$.
10. Max. $23\frac{1}{4}$ when $x = \frac{3}{2}$; min. 23 when $x = 2$.
11. Max. 2 when $x = 3$; min. $-$ 2 when $x = 1$.
12. 25, 25.
13. $\frac{1}{2}$.
14. 1.
15. $h = d$.
16. Depth = 0·816 diam.; Breadth = 0·577 diam.
17. 98·3 yds. square.
18. 9·21 knots.
19. $v = 92{\cdot}4$, H.P. = 24·6.
20. $r = 5{\cdot}29$, N = 17·46.
21. 3.
22. 15 ft. by $7\frac{1}{2}$ ft.

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